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UNIVERSITY OF ILLINOIS

EDWARD G. BRENNER COLLEGE

THE DEVELOPMENT AND EVALUATION OF TEXT MATERIALS
COVERING TOPICS FROM FORMAL LOGIC AS RELATED
TO THE TEACHING OF HIGH SCHOOL MATHEMATICS

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THESIS

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CHAPTER I INTRODUCTION

A secondary school mathematics teacher in 1963-64 is faced with a variety of new programs from which to develop a curriculum for his school. Although each program has unique features, all share common elements. One of these common elements is an emphasis on logical deduction (3, 27), (9, 95).¹ While the use of deductive reasoning is common to all the newer programs, the emphasis in the different curricula on the nature of deductive reasoning ranges from an apparent assumption by the authors of Mathematics 9, produced by the Ontario Mathematics Commission, that "some acquaintance with the nature of proof has been had earlier than grade 9" (5, 30) to an appendix entitled "Rules of Reasoning" in Unit 6, High School Mathematics, produced by the University of Illinois Committee on School Mathematics (13, 357-400). This appendix includes a discussion of some of the rules of reasoning and principles of logic used in proving theorems together with illustrations of their use in proofs of algebra theorems.

In 1955 the Commission on Mathematics was established by the College Entrance Examination Board. The members of the Commission were "asked to review the existing secondary school mathematics curriculum, and to make recommendations for its modernization, modification, and improvement" (4 xi).

¹ References will be indicated by one of two forms, '(x)', or '(x, y)', where 'x' refers to an entry in the bibliography and 'y' to the page or pages to which reference is made.

The major proposals of the Commission are outlined in nine points, one of which is:

2. Understanding of the nature and role of deductive reasoning--in algebra, as well as in geometry. (4, 33)

This point is further emphasized by the statement:

The Commission is firmly of the opinion that deductive reasoning should be taught in all courses in school mathematics and not in geometry courses alone. (4, 22)

The Problem

Students in secondary school mathematics classes often are required to prove theorems. For many students a "proof" is thought of as "what convinces me". When students are asked to prove, for example, the sentence:

$$2x + 3x = 5x$$

they often rebel, since the sentence is "obviously true". In his book,

Introduction to Logic, Suppes states the problem this way:

You should not be thrown intellectually off balance by the seemingly trivial content of some of the theorems. You may in such cases be inclined to say to yourself, This is so obvious it is silly to ask me to prove it. But if you say this you are confusing the obviousness of the truth of the theorem with the obviousness of the proof that the theorem is a logical consequence of the axioms. (11, 132)

Use of the deductive method in algebra and geometry is probably not sufficient to lead students to an awareness of the essential nature of the deductive method, that is, that certain sentences are logical consequences of other sentences solely on the basis of accepted rules of reasoning and the form of the sentences. In the appendix to Unit 6, High School Mathematics, the authors state:

... a proof shows how its conclusion follows, step-wise, from its premises, by the application of principles of logic. So, one can scarcely understand the nature of proof unless he is acquainted with at least some of the logical principles which justify his inferring of later steps in the proof from earlier ones. (13, TC [6-357a])

One way to become acquainted with these principles is by discussing them as they arise in proofs in algebra and geometry. However, this discussion usually takes place in a context where the emphasis is on the "facts" of mathematics rather than on the nature of deductive reasoning. Another way to become acquainted with these principles is by studying text materials in which the major emphasis is on the rules of reasoning themselves. Such text materials have been written but have been limited in use to undergraduate or graduate students.

There now exists a large number of secondary school students who are called upon to use the deductive method to a greater extent than before. Such students are ready for a discussion of the nature of deductive reasoning. Hence, one aspect of the problem is the lack of text material in formal logic which is suitable for these students.

Although the need for text material for these students seems clear, the appropriate level for the use of such materials is open to discussion. In the Twenty-Third Yearbook of the National Council of Teachers of Mathematics, Carl Allendoefer states:

Since the deductive method is an essential part of modern mathematical thinking, the teacher should use every opportunity to illustrate it in every aspect of her work. Illustration, however, is probably not enough to teach the students the essential structure of a deductive system. At some stage in the high school mathematics curriculum there should be a serious discussion of deductive systems per se, and later applications of this to mathematics and to nonmathematical situations should be used to reinforce the understanding of the students about deductive methods. Perhaps the tenth grade is the place for this, but no firm statement of this kind should be made until more experimental teaching has been carried out. (1, 66)

The National Council of Teachers of Mathematics Committee on the Analysis of Experimental Programs states that among the issues to be considered in evaluating any mathematics program is the issue of the place of proof. The report raises these questions:

At what level should proof be introduced and with what degree of rigor? How rapidly should a student be led to make proofs independently? At what level should he be aware of what mathematical proof is? (5, 3)

The problem then is two-fold; to provide students with text materials in formal logic, and to determine the effectiveness of this material. The study herein described is designed to produce both a text in formal logic and data which may be used to evaluate its effectiveness for particular groups of students.

Method

The present study was designed to provide a text in formal logic for secondary school students and to provide teachers with data which they might use as one set of criteria for deciding whether the text is appropriate for their students. These two objectives suggested the procedure outlined below:

- (1) Initial preparation of materials
- (2) Trial and revision of materials
- (3) Preparation of a final test
- (4) Selection of a sample to study the material
- (5) Analysis and summary of test data

It was decided that a complete treatment of formal logic was not appropriate for secondary school students. Topics from formal logic to be included in the text were selected on the basis of their relevance to high school mathematics as presented in the newer programs. The units produced by the

² For a description of the VICBM program, see Hale (6).

University of Illinois Committee on School Mathematics (UICSM) were chosen as an example of the newer programs.² An analysis of these units led to the selection of the topics discussed in detail in Chapter II of this paper.

A preliminary version of the text was tried with students at Barrington High School, Barrington, Illinois. This trial was designed to test the hypothesis that the study of the logic text produced students with increased knowledge of formal logic. Unless such a hypothesis is tenable, there is little justification for requiring students to study the text.

The trial group at Barrington also provided data for revision of the text and the final test. An analysis of workbooks and answer sheets suggested the need for changes in the text and the test and provided information about the difficulty of test items. Such information was used as a basis for arranging test items in order of increasing difficulty.

The Sample

One objective of the study was to provide teachers with information about the level of post-study performance of well-defined groups of students. This information could be used by teachers who are considering the use of the logic text by a particular student or group of students. As Allendoefer indicated, there is little objective evidence which shows that the tenth grade is the best place to study formal logic. The sample was selected to include groups

- (a) from grade levels 8 through 12
- (b) with different mathematical backgrounds
- (c) with intelligence test scores from different tests

These differing groups were selected to increase the probability that a teacher would find information pertinent to the particular student or group of students for whom he is considering the text.

² For a description of the UICSM program, See Hale (8).

Topics from beginning algebra were used in the logic text to provide applications of the rules of reasoning. Thus, a student was "eligible" to study the programed materials if he

- (1) was then studying Unit 2 of UICSM High School Mathematics,
- or (2) was then enrolled in "regular" first year algebra,
- or (3) was then enrolled in some mathematics course beyond first year algebra.

A student who met one of these conditions would have had some experience with the use of numerical variables and some knowledge of algebraic manipulation. He would thus be in a position to concentrate on the reasoning used in proving theorems instead of worrying about the subject matter of these theorems.

The sample consisted of 502 subjects from classes in four public schools and one Catholic parochial school. The composition of the sample by schools and grade level is given in Table 1. Pascack Valley High School, as indicated in the table, furnished 76 ninth grade students, 15 tenth grade students, and 48 eleventh grade students.

Central Catholic High School
Portland, Oregon

	37	25	72	44	
Totals	26	153	118	125	48

CHAPTER II

INSTRUCTIONAL MATERIALS

TABLE 1

COMPOSITION OF THE SAMPLE

BY SCHOOLS AND GRADE LEVEL

School	Grade	8	9	10	11	12
Jefferson High School Denver, Colorado				12		
Wheat Ridge Junior High Wheat Ridge, Colorado			37			
Pascack Valley Regional Hillsdale, New Jersey		76	15	48		
Boulder City Jr. - Sr. High Boulder City, Nevada		26	36	62		
Central Catholic High School Portland, Oregon			39	29	78	44
	Totals	26	188	118	126	44

Subject Matter

The analysis of the DICM units indicated that students working with these materials would need to understand rules of reasoning for the following types of statements:

- Universal generalizations
- Equations
- Conditional sentences
- Denial sentences
- Conjunctions
- Alternatives
- Biconditionals

A summary of the rules of reasoning developed in the text is given on pages 43-45.

CHAPTER II

INSTRUCTIONAL MATERIALS

The units produced by the University of Illinois Committee on School Mathematics (UICSM) were chosen as an example of the newer programs. Topics from formal logic were selected for development in the text on the basis of their relevance to the types of reasoning used in these units. The appendix to Unit 6, High School Mathematics "furnishes an introduction to some of the more commonly used logical principles, and contains illustrations of their use in proofs of theorems from algebra" (13, TC [6-357]a). The programed text written for this study followed closely the development of logical principles as given in the appendix to Unit 6. However, the first portion of the programed text included a development of the use and meaning of universal generalizations, a topic not treated in the appendix. This material was included so that students who had not studied the first 4 units of the UICSM materials would not be handicapped when reading the remainder of the text.

Subject Matter

An analysis of the UICSM units indicated that students working with these materials would need to understand rules of reasoning for the following types of sentences:

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- Biconditionals

A summary of the rules of reasoning developed in the text is given on pages 43-45.

Universal Generalizations and Equations

In Unit 1 of UICSM High School Mathematics, students become acquainted with the arithmetic of the real numbers and become convinced of the truth of some basic principles about operations with real numbers (the commutative principle for addition, the associative principle for multiplication, etc.). In Unit 2 students learn how to state these basic principles as universal generalizations [$\forall x \forall y x + y = y + x$, $\forall x \forall y \forall z (xy)z = x(yz)$] and also learn how to derive other generalizations from the basic principles.

The authors of the UICSM materials state in Unit 2:

The procedure of taking a known subject matter [here, the arithmetic of the real numbers] and organizing it deductively, by choosing some true statements from it as basic principles [or postulates] and deriving others [theorems] from them, is a common one in mathematics, and is even more common in applications of mathematics. (12, TC[2-60]a)

Much of the work in mathematics requires skill in manipulating expressions; for example, one procedure for solving the equation ' $3x - 7 + 2x = 3$ ' is to transform it to an equivalent equation whose root is obvious. ' $5x - 7 = 3$ ' is such an equation. In order to prove that these equations are equivalent, it is necessary to show that the expressions ' $3x - 7 + 2x$ ' and ' $5x - 7$ ' are equivalent, and in order to do this, one needs to show that the universal generalization:

$$(*) \forall x 3x - 7 + 2x = 5x - 7$$

is a theorem. Since the sentence $(*) \forall x 3x - 7 + 2x = 5x - 7$ and the basic principles are universal generalizations, students who wish to derive $(*)$ from the basic principles need rules of reasoning which enable them to justify the use of universal generalizations in derivations. Two rules of reasoning, Universal Instantiation (UI) and the Test Pattern Principle (TPP),

together explain the meaning of universal generalizations. These two rules together with the Replacement Rule for Equations (RRE) are developed by showing their use in establishing that certain sentences are theorems.

Here is a proof that the universal generalization ' $\forall y \ 0 + y = y$ ' is a theorem:

- rules (1) $a + 0 = 0 + a$ (2) $\forall x \forall y \ x + y = y + x$ of these rules UI
 reason (2) is $a + 0 = a$ after we study $\forall x \ x + 0 = x$ studied at least on UI
 of math (3) $0 + a = a$ similar rule (familiar from use of (1), (2); RRE) is Mod
 is Mod (4) $\forall y \ 0 + y = y$ the rule which justifies inferring (1) - (3); TPP on

Just as Universal Instantiation and the Test Pattern Principle explain the use of universal generalizations, the Replacement Rule for Equations (RRE) together with the Law of Identity (LI) explain the use of equation sentences. These four rules of reasoning (UI, TPP, RRE, and LI) are sufficient for simple proofs such as the one given above. For additional information about the development and use of these rules, see pages 39-109 of Rules of Reasoning given in Appendix A.

Conditional Sentences

Although equation sentences and universal generalizations are an important part of a deductive system, many theorems in algebra and geometry are universal generalizations whose instances are conditional sentences, that is, sentences of the form:

if _____ then _____

For example,

$$\forall x \forall y \text{ if } x + y = 0 \text{ then } -x = y$$

If the opposite sides of a quadrilateral are congruent,

then the quadrilateral is a parallelogram.

are two such theorems.

Just as there are two basic rules of reasoning for universal generalizations, one for justifying inferences from universal generalizations (Universal Instantiation) and one for arriving at conclusions which are universal generalizations (Test Pattern Principle), there are corresponding rules for reasoning with conditional sentences. One of these rules of reasoning is quite familiar to students who have studied at least one year of mathematics. The familiar rule (familiar from use if not known by name) is Modus Ponens (MP), the rule which justifies inferring a conclusion from two premisses, one of which is a conditional sentence. Here is a proof in which one step is justified by Modus Ponens:

- | | | |
|------------------------------------|--|-----|
| (1) $a + -a = 0$ | $\forall x \ x + -x = 0$ | UI |
| (2) if $a + -a = 0$ then $a = --a$ | $\forall x \forall y \text{ if } x + y = 0 \text{ then } x = -y$ | UI |
| (3) $a = --a$ | (1), (2); | MP |
| (4) $\forall x \ x = --x$ | (1) - (3); | TPP |

The other basic rule for conditional sentences, the one used for arriving at conclusions which are conditional sentences is not so well known or used. The rule is the Discharge Rule (D):

If the consequent of a conditional sentence is a consequence of the antecedent of the conditional sentence, and other premisses, then the conditional sentence is a consequence of the other premisses alone.

The type of inference justified by the Discharge Rule is usually omitted in proofs shown in standard textbooks. For example, in Plane Geometry by Welchons, Krickenberg, and Pearson (17, 207) there is a proof of the theorem:

If the opposite sides of a quadrilateral are equal,
the figure is a parallelogram.

The proof is consistent with the earlier definition of proof:


The proof consists of a series of deductions beginning with the facts in the hypothesis and ending with the conclusion. (17, 68)


The last line of the proof is:

ABCD is a 

One of the premisses in the proof is:

(*) $AD = BC$ and $AB = DC$

The premiss (*) is equivalent to the antecedent of an instance of the theorem and the conclusion 'ABCD is a ' is the consequent of an instance of the theorem. Thus, a sentence justified by the Discharge Rule would be:

if $AD = BC$ and $AB = DC$ then ABCD is a .

From the authors' definition of proof, there is no need for this sentence.

However, since the theorem is stated as a conditional sentence

(if _____ then _____), it seems that a derivation of the theorem should end with a conditional sentence.

In the logic text used in this study, a theorem is defined as a statement which is the conclusion of a valid derivation whose only premisses are generalizations which we have accepted. Here is a proof that the statement:

$$\forall x \forall y \forall z \text{ if } x = y \text{ then } x + z = y + z$$

is a theorem:

(1) $a + c = a + c$	$\forall x \ x = x$	UI
(2) $a = b$	Assumption*	
(3) $a + c = b + c$		(1), (2); RRE
(4) if $a = b$ then $a + c = b + c$		C(3); D(2)*
(5) $\forall x \forall y \forall z \text{ if } x = y \text{ then } x + z = y + z$		(1)-(4); TPP

The '*'s show the use of the Discharge Rule. Having derived the consequent of the conditional 'if $a = b$ then $a + c = b + c$ ' from the antecedent ' $a = b$ ', the discharge rule justifies our inferring the conditional sentence itself and lines (1) - (4) show that the conditional sentence is a consequence of the other premiss ' $\forall x \ x = x$ '. The premiss ' $a = b$ ' used to derive ' $a + c = b + c$ ' has been discharged. Then by using the Test Pattern Principle we complete a derivation which shows that the statement ' $\forall x \forall y \forall z \text{ if } x = y \text{ then } x + z = y + z$ ' is a theorem.

It is interesting to note here that the rules of reasoning established earlier (UI, RRE, and TPP) were used in the derivation in addition to the new Discharge Rule. Without these earlier rules, it would be difficult to write derivations in which the Discharge Rule is used.

Denial Sentences, Modus Tollens, and Contraposition

At times it is convenient to prove a theorem by use of contraposition. For example, here is a proof of the theorem:

- $\forall x \forall y$ if $x + y \neq 0$ then $y \neq -x$
- (1) $a + -a = 0$ $\forall x$ $x + -x = 0$ UI
- (2) $b = -a$ Assumption*
- (3) $a + b = 0$ (2), (1); RRE
- (4) if $b = -a$ then $a + b = 0$ C(3), D(2)*
- (5) if $a + b \neq 0$ then $b \neq -a$ (4); Cp
- (6) $\forall x \forall y$ if $x + y \neq 0$ then $y \neq -x$ (1)-(6); TPP

The 'Cp' on line 5 refers to one of the rules of contraposition developed in the text. The four rules of contraposition useful in proving theorems are given by the inference schemes:

- (I) $\frac{\text{if } p \text{ then } q}{\text{if not } q \text{ then not } p}$ (II) $\frac{\text{if not } p \text{ then not } q}{\text{if } q \text{ then } p}$
- (III) $\frac{\text{if not } p \text{ then } q}{\text{if not } q \text{ then } p}$ (IV) $\frac{\text{if } p \text{ then not } q}{\text{if } q \text{ then not } p}$

The development of these rules was our first opportunity to show how rules of reasoning which we accept as basic can be used to justify other rules of reasoning as well as to justify steps in a derivation. Here is an inference scheme which justifies one of the rules of contraposition:

$$\frac{\frac{\text{if } p \text{ then not } q \quad \frac{q}{\text{not (not } q)} \text{ DD}}{\text{not } p} \text{ MT}}{\text{if } q \text{ then not } p} \text{ C, D*}$$

Since denial sentences [sentences of the form 'not (_ _)'] occur in inferences justified by Contraposition, a discussion of such sentences preceded the rules of contraposition.

Modus Tollens, a rule of reasoning used to justify inferences in which both denial sentences and conditional sentences occur, was another rule of reasoning developed in this section. Modus Tollens justifies inferring the denial of the antecedent of a conditional from the conditional together with the denial of the consequent of the conditional. Schematically,

For alternations the basic

$$\frac{\text{if } p \text{ then } q \quad \text{not } q}{\text{not } p}$$

is a valid inference scheme

Although the inference schemes for Modus Tollens and Contraposition refer to sentences of the form 'not (_ _)', sentences such as ' $a \neq 0$ ' more often occur in mathematics than do sentences such as ' $\text{not } (a = 0)$ '. Hence, we modified the use of these rules to permit justification of inferences such as:

if $a = 0$ then $ab = 0$ $ab \neq 0$
 $\hline a \neq 0$

For additional information about denial sentences, Modus Tollens, and Contraposition, see pages 177-206 of Rules of Reasoning.

Conjunctions and Alternations

Conjunctions, sentences of the form:

_____ and _____
 and alternations, sentences of the form:

_____ or _____
 often occur in mathematical sentences. For example:

$\forall x \forall y$ if $xy = 0$ then $(x = 0 \text{ or } y = 0)$

$\forall x \forall y \forall z$ if $x > y$ and $y > z$ then $x > z$

are two very useful theorems. Conjunctions also occur when discussing the concept of a counter-instance of a universal generalization. A counter-instance

of a universal generalization is a sentence which is true and which is a contradictory of an instance of the generalization. The basic rules of reasoning for conjunctions are:

$$(I) \frac{p \quad q}{p \text{ and } q}$$

$$(II) \frac{p \text{ and } q}{p}$$

$$(III) \frac{p \text{ and } q}{q}$$

For alternations the basic rules of reasoning are:

$$(I) \frac{p}{p \text{ or } q}$$

$$(II) \frac{q}{p \text{ or } q}$$

$$(III) \frac{p \text{ or } q \quad \text{not } q}{p}$$

$$(IV) \frac{p \text{ or } q \quad \text{not } p}{q}$$

$$(V) \frac{p \text{ or } q \quad \text{if } p \text{ then } r \quad \text{if } q \text{ then } r}{r}$$

A connection between conjunctions and alternations is shown by the inference schemes:

$$(I) \frac{\text{not } (p \text{ and } q)}{\text{not } p \text{ or not } q}$$

$$(II) \frac{\text{not } (p \text{ or } q)}{\text{not } p \text{ and not } q}$$

For additional information about the use of conjunctions and alternations, see pages 208-237 of Rules of Reasoning.

Biconditionals

A biconditional, a sentence of the form:

_____ if and only if _____

is considered as an abbreviation of the conjunction of two conditionals:

if _____ then _____

if _____ then _____

Since a biconditional can be unabbreviated to a conjunction, we can use our inference patterns for conjunctions when dealing with biconditionals.

A new rule of reasoning which is useful when reasoning from biconditionals is the Replacement Rule for Biconditionals:

Given a biconditional and another sentence, if one side of the biconditional sentence is replaced by the other side somewhere in the other sentence, the new sentence thus obtained is a consequence of the given sentences.

Here is an example of how the replacement rule for biconditionals is used in justifying a method of solving quadratic equations. The quadratic equation:

$$(1) \quad 8x^2 - 10x - 3 = 0$$

can be transformed to the equivalent sentence:

$$(2) \quad 2x - 3 = 0 \text{ or } 4x + 1 = 0$$

Using two equation transformation principles, the biconditionals:

$$(a) \quad 2x - 3 = 0 \text{ if and only if } x = \frac{3}{2}$$

$$(b) \quad 4x + 1 = 0 \text{ if and only if } x = -\frac{1}{4}$$

can be derived. The replacement rule for biconditionals tells us that the sentence:

$$(2.1) \quad x = \frac{3}{2} \text{ or } 4x + 1 = 0$$

is a consequence of (a) and (2) and that

$$(3) \quad x = \frac{3}{2} \text{ or } x = -\frac{1}{4}$$

is a consequence of (b) and (2.1). Also, (2.1) is a consequence of (3) and (b),

and (2) is a consequence of (2.1) and (a). Thus, (2) and (3) are equivalent

sentences. The roots of (3) are $\frac{3}{2}$ and $-\frac{1}{4}$. Since (3) is equivalent to (2) and (2) is equivalent to (1), we know that the roots of (1) are $\frac{3}{2}$ and $-\frac{1}{4}$.

$$\begin{array}{ccc} (1) & & (2) \\ \text{John is poor} & \text{if and only if} & \text{John is poor then John is happy} \\ & & \text{John is happy} \\ & & (3) \end{array}$$

The Nature of Proof

In addition to rules of reasoning, a discussion of the nature of proof and the relation between validity and truth is included in the text. The UICSM authors state:

There is a further step, peculiar to mathematics, in which one forgets entirely the 'known subject matter' and, considering the postulates and theorems merely as sentences in an uninterpreted language, concentrates his attention on the logical connections among these sentences. (TC [2-60]a)

The terms

premiss	conclusion
derivation	theorem

are useful in explaining the logical connections among sentences. These terms are introduced in discussions of particular proofs and followed by examples of other proofs, some of which the student must complete. He is also asked to write and justify complete proofs of his own. The main objective of this section is summarized by the concluding remarks on page 122 of Rules of Reasoning:

Note well that the purpose of proof is not to convince you of the 'correctness' of the generalization. In proving that the generalization is a theorem, we are not trying to convince you that the generalization is true. [You are probably as sure of this generalization as you are of the generalizations which we took as principles.] We prove that ' $\forall x \forall y \quad x(y + 1) = xy + x$ ' is a theorem by showing that it is a logical consequence of our principles.

Validity and Truth

Our grounds for accepting an inference as valid has nothing to do with truth. We accept Modus Ponens-type inferences as valid because of the meaning which we intend the phrase 'if _____ then _____' to have. The validity of the inference:

(1)	(2)
John is poor	if John is poor then John is happy
<hr/>	
	John is happy
	(3)

is a consequence solely of the fact that (2) is a conditional, (1) is its antecedent, and (3) its consequent. Which, if any, of the three statements are true and which are false has no bearing on the validity of the inference. That is, the validity of the inference depends only on the form of the sentences, regardless of their truth or falsity. Similar remarks apply to the other rules of reasoning.

Hence, the truth of premisses and conclusion does not guarantee the validity of an inference. Also, the validity of an inference does not guarantee the truth of the premisses and conclusion. In fact, in proofs by contradiction one often begins with a premiss which is believed to be false. One difficulty students have with proof by contradiction is probably due to the failure to distinguish between accepting a statement as a premiss and accepting a statement as being true. A derivation with a given statement as a premiss and a false statement as conclusion is often a basis for deciding that the given statement, although taken as a premiss, is in fact false.

The connection between truth and validity is expressed in the rule:

- (i) If some consequence of a set of premisses is false, and the derivation is valid, then at least one of the premisses is false.

Rule (i) together with

- (ii) The truth value of a complex statement depends only on the truth value of its components

- (iii) Not all statements are true

and our basic rules of inference are sufficient for deriving the usual "truth-tables" for conditionals, denial-sentences, conjunctions, and

alternations. These truth tables can then be used to suggest the equivalence of, for example, sentences of the form:

(a) if p then q

(b) not p or q

but the fact of equivalence must be established by showing that (a) is a logical consequence of (b) and that (b) is a logical consequence of (a).

Derived Rules of Inference

The basic rules of inference discussed herein may be used to justify other useful rules of inference. For example, here is a derivation of the rule of reasoning called importation:

	*	
	p and q	
p and q	p	if p then [if q then r]
q	if q then r	
<hr/>		r
<hr/>		if (p and q) then r
		*

There is no attempt made in the book Rules of Reasoning to develop and justify each type of reasoning that a mathematics student may wish to use in proving theorems. However, as shown, the basic rules which are established are useful for justifying other rules as well as for justifying particular inferences in a proof. Given the foundation provided by the study of the text, a student is in a position to use the basic rules when writing proofs usually called for in secondary school mathematics and to justify other types of reasoning which he may elect to try.

Initial Preparation and Trial of Materials

After selection of the subject matter for the book, decisions were made regarding the ordering of the topics and the method of presentation.

Order of Presentation

The subject matter can be separated into three categories

- A) Rules of reasoning for simple sentences; universal generalizations and equations;
- B) Rules of reasoning for complex sentences; denials, conditionals, conjunctions, alternations and biconditionals;
- C) Subject matter related to the concepts of validity and truth, and the nature of proof.

The order of presentation of the topics was based on their relative complexity. Some decisions regarding order were dictated by the kind of knowledge prerequisite for the understanding of the development of a particular topic. For example, rules of reasoning for denial sentences are used in developing the rules of contraposition. Hence, denial sentences preceded contraposition.

Since simple sentences are used as components of complex sentences, it is desirable to discuss simple sentences first. Universal generalizations are used to state the basic principles for operations with real numbers and are also used as premisses in derivations of other generalizations about those operations. Hence, the discussion of universal generalizations and rules of reasoning for inferences involving universal generalizations was the first topic presented. Although many of the computations in arithmetic can be justified by the basic principles alone, many "shortcuts" are not immediate consequences of these principles.

For example, most people "just know" that ' $(13 + 5) - 5$ ' and ' 13 ' are equivalent numerals. But the sentence:

$$(13 + 5) - 5 = 13$$

is not an immediate consequence of the basic principles. Since one of the objectives of the unit was to show the reasoning used to justify shortcuts as

consequences of the basic principles, various theorems which treated operations with real numbers were discussed next. Here is a derivation of the sentence ' $(13 + 5) - 5 = 13$ ':

- (1) $(13 + 5) - 5 = (13 + 5) + -5$ $\forall x \forall y x - y = x + -y$ UI
- (2) $(13 + 5) + -5 = 13 + (5 + -5)$ $\forall x \forall y \forall z (x + y) + z = x + (y + z)$ UI
- (3) $(13 + 5) - 5 = 13 + (5 + -5)$ (1), (2); RRE
- (4) $5 + -5 = 0$ $\forall x x + -x = 0$ UI
- (5) $(13 + 5) - 5 = 13 + 0$ (3), (4); RRE
- (6) $13 + 0 = 13$ $\forall x x + 0 = x$ UI
- (7) $(13 + 5) - 5 = 13$ (5), (6); RRE

Equations occur throughout this derivation and the Replacement Rule for Equations (RRE) is used to justify lines (3), (5), and (7). Similar derivations could be written for sentences such as:

$$(A) (8 + 2) - 2 = 8$$

$$(B) (19 + 7) - 7 = 19$$

$$(C) (36 + 8) - 8 = 36$$

These sentences show a pattern which can be summarized by the universal generalization:

$$(*) \forall x \forall y (x + y) - y = x$$

Thus, if one shows that $(*)$ is a theorem, it could then be used as a justification for each of the sentences (A), (B), and (C). In order to show that $(*)$ and other universal generalizations which justify computational shortcuts are theorems, rules of reasoning are needed which enable one to handle equations and to write a derivation whose conclusion is a universal generalization. This suggested that rules of reasoning for equations together with the Test Pattern Principle for universal generalizations be presented next in the text.

Other shortcuts, such as the one used to transform the expression $-(a + b)$ to $-a + -b$, can be easily justified by the theorem:

$$\forall x \forall y \text{ if } x + y = 0 \text{ then } -y = x$$

Many other theorems have as instances sentences of the 'if _____ then _____' form. Since conditional sentences are complex sentences with simple sentences as components, development of rules of reasoning for conditionals followed the development of rules for the simple sentences. The development of rules of reasoning for conditional sentences, Modus Tollens, Modus Ponens, and Contraposition required the treatment of denial sentences as an integral part.

After denial sentences had been discussed, it was possible to treat the concept of counter-instance. Students often propose as justifications for their shortcuts a universal generalization which is not a theorem. A single counter-instance is enough to show that a generalization is not a theorem. The discussion of a counter-instance as a sentence which is true and which is a contradictory of an instance of the generalization led naturally to the treatment of conjunctions. Closely related in form to conjunctions are alternations. The denial of a conjunction of the form:

$p \text{ and } q$

is equivalent to an alternation of the form:

$\text{not } p \text{ or not } q$

Thus, these two topics were considered together. Since a biconditional is an abbreviation of the conjunction of a conditional and its converse, rules of reasoning for biconditionals followed those for conjunctions.

In summary, the ordering of topics was based on their complexity and the kind of knowledge prerequisite for their develop-

Method of Presentation

The design of the study required that the text material be suitable for students with different mathematical backgrounds and different levels of mathematical development. A program format was chosen so that students could cover the topics at individual rates of speed. Teachers had indicated to the author that they could use a logic book which was suitable for individual study. Such a book could be given to selected students from a class without disrupting the classroom routine. Although all students from a particular class participated in the study, future use of the text might be limited to only a portion of a class. A teacher might decide, on the basis of data provided by this study and interest shown by students, to have some of the students in a class study the formal logic and decide that others in the same class were not yet ready for these topics. The availability of a programed text would enable this teacher to provide these students with material they could study outside the regular classroom environment.

Decisions regarding details of presentation were influenced by the author's close association with the Programed Instruction Project conducted by members of the UICSM staff.³ Each rule of reasoning was developed in a separate linear sequence. However, there was some overlap in these sequences when the end of one sequence was used as an introduction to the next topic. As indicated in the section on order of presentation, some topics were prerequisites for others. These prerequisite topics often reappear in

³For a description of the Programed Instruction Project, see Wills (18).

later sections, for example, work with denial sentences occurs in the sequence for contraposition as well as in a separate section of its own.

Each sequence consisted of three parts presented in the following order:

1. A series of pages designed to lead the reader to a non-verbal awareness of the concept or generalization;
2. One or more pages in which the student could see the rule formally stated;
3. Additional pages requiring application of the rule.

See pages 69 through 82 of Rules of Reasoning for the development of the Replacement Rule for Equations. Pages 69 through 75 make up Part 1 of the sequence, pages 75a and 75 make up Part 2, and Part 3 consists of pages 75 through 82.

A typical page of the book consisted of three parts; (1) answers for the exercises appearing on the preceding page, (2) discussion of previous exercises or presentation of new ideas, and (3) a set of exercises to be answered in the workbook. Each student in the study was provided with a workbook in which to record his answers. Although the workbooks were used in the study, the text was written so that it could be used without a formal workbook in which to record answers. Ordinary notebook paper would be sufficient. Workbooks were provided for students participating in the study so that student responses would be readily available as one source of information on which to base a revision of the text.

⁴ VICSUM Staff, Activities in the field of self-instruction. August, 1963. (An unpublished information bulletin.)

The linear technique used in writing the text is based on that used by those writing for the Programed Instruction Project of the UICSM.

Much flexibility derives from the fact that our use of a linear technique has not been restricted to the presentation of just small bits of information at a time. Instead, a full-page format is usually used, and there are sometimes lengthy discussions of previous problems or of new ideas. There are frequent illustrations. With occasional exceptions, we have up to ten problems on a page, all to be done before checking any answers. In a certain sense, the variety of situations and problems encountered by each student approximates what he encounters in a carefully conducted UICSM class.⁴

The series of pages designed to lead the reader to a non-verbal awareness of the concept or rule include several "prover" exercises. These exercises provide the student, and the person who examines the student's answers, with evidence that he has, or perhaps has not, become aware of the concept or rule. The other exercises in a sequence are designed to correct any misconceptions a student might have concerning the rule. A branching program based on this text might be written with the material after the first prover exercises being used in the branches. A detailed analysis of students' answers and interviews with students would provide information about the appropriate places for these branches. The present arrangement provides the same sequence for each student, regardless of how soon he discovers the rule. However, since a student may progress at his own rate, the pages which are superfluous for one student will be covered very quickly by that student and also serve as additional reinforcement.

⁴ UICSM Staff. Activities in the field of self-instruction. August, 1963. [An unpublished information bulletin.]

Trial of Materials

A preliminary version of the programed text was written in the school year 1962-63. The trial version was studied by students in grades 9, 10, and 11 at Barrington High School, Barrington, Illinois. At this same time, a 60-item test was written to cover the objectives of the text. A copy of the revised test is given in Appendix B. The test was given to the Barrington students who had studied the text and to a second group of Barrington students who had not studied the text. Each group contained a ninth grade class of students who were studying the UICSM Unit 3, a ninth grade class studying "regular" algebra, a ninth grade class studying UICSM Unit 6, two tenth grade classes studying UICSM Unit 6, two tenth grade classes studying regular geometry, and one eleventh grade class studying regular algebra. The test mean for those who had studied the material was 28.9; the mean for those who had not studied the material was 20.5. A t-test was used to test the significance of this difference. The difference is significant at the .01 level. On the basis of these results, it was decided that the study of the programed materials made a difference in knowledge of formal logic.

Revision of Materials

Revision of the materials was based on an analysis of worksheets and test answer sheets from the Barrington students. The test data were analyzed according to the method presented in Ross and Stanley, Measurement in Today's Schools (10, 153-156), using answer sheets from the upper and lower 27 per cent. Test items for which $W_L + W_H$ was greater than the number which indicated that 84 per cent or more of the students did not know the answer were selected for special examination (W_L is the number in the lower 27 per cent who got the item wrong). These test items were examined for possible ambiguities and the related pages in the text were revised by

rewriting and the addition of new material. An analysis of the worksheets showed that some topics had been treated too extensively. For example, the first 29 pages of the trial version were written to teach the students that false statements, as well as true statements, could be universal generalizations. Students commented on worksheets that this was clear to them several pages before the "prover" page was presented. The percentage of correct responses to earlier questions also indicated they had learned this point and additional pages were superfluous. Hence, these pages were revised to include more practice in translating from ordinary English to the formal language of mathematics and more emphasis on existential generalizations. Other text changes were made at the suggestion of the author's colleagues at the University of Illinois.

Test

An achievement test was written to sample understanding of the subject matter as presented in the program. A preliminary version of the test was given to a group of students in University High School, Urbana, Illinois, who had studied the appendix to Unit 6, High School Mathematics (13, 357-400). The main objective of this trial was to identify items which were ambiguous and to obtain information about the difficulty of items. The data about difficulty were used to arrange items in order of increasing difficulty on the next version of the test. In addition to statistical analysis, student interviews were used to obtain this information.

A penultimate version of the test was then prepared for use with the Barrington High School students. This version of the test had the same form as the final test as given in Appendix B. The test consisted of three parts; Part I was designed to test knowledge of terminology and conventions, Part II to test application, and Part III to test evaluation.

The categories of knowledge, application and evaluation were taken from the Taxonomy of Educational Objectives edited by Benjamin Bloom (2). These three categories represent three types of student behavior which should be altered as a result of studying the text. The definitions of the three types of behavior given in the Taxonomy are:

For our taxonomy purposes we are defining knowledge as little more than remembering the idea or phenomenon in a form very close to that in which it was originally encountered. (2, 28-29)

Given a problem new to the student, he will apply the appropriate abstraction without having to be prompted as to which abstraction is correct or without having to be shown how to use it in that situation. (2, 120)

Evaluation is defined as the making of judgments about the value, for some purpose, of ideas, works, solutions, methods, etc. (2, 185)

The test was examined for content validity by listing items according to subject matter category. This listing is given in Table 2. The high proportion of items dealing with conditionals is consistent with the proportion of the text devoted to the development of rules of reasoning related to conditionals. A difficulty index was computed for each item from the formula:

$$\text{Difficulty index} = \frac{100 R}{N}$$

where R is the number of students who got the item right and N is the number of students who took the test. Information from Table 46 in Measurement in Today's Schools by Ross and Stanley (10, 448) was used to evaluate the discriminatory power of each item.

A final version of the test was then prepared by using items from the trial version. Those items with difficulty indices below .16 were revised as were items which were not sufficiently discriminating. The items in each part were then arranged in order of increasing difficulty.

Data obtained from the students who studied the final version of the test and took the final test were analyzed by using the scores of students with test scores in the upper 27 per cent and in the lower 27 per cent of the test scores. Each item from the final test

TABLE 2
TEST ITEMS CLASSIFIED BY SUBJECT MATTER

Subject Matter	Part I	Part II	Part III
Generalizations	1, 6, 17	9	12, 2, 18, 20
Equations		3, 4, 8	7
Conditionals	4, 8, 9, 10, 11, 13, 14, 15	1, 5, 6, 10, 11, 13, 14, 15	2, 4, 5, 6, 9 10, 12, 17, 19
Denials	16	2, 20	3, 6, 11, 16, 20
Conjunctions	16		1, 11, 16
Alternations		7, 12, 19	8, 13, 15
Biconditionals	19	17	14
Theorem	2, 20		
Contraposition	3, 5	16	
Inference	7, 12, 18		

Data obtained from the students who studied the final version of the text and took the final test were analysed by using the scores of students with test scores in the upper 27 per cent or in the lower 27 per cent of the test score distribution. A difficulty index was computed for each item from the formula:

$$\text{Difficulty index} = \frac{100 [R_L + R_H]}{2n}$$

where R_L is the number of students in the lower 27 per cent who got the item right, R_H is the number of students in the upper 27 per cent who got the item right, and n is the number of students in the lower 27 per cent. A discrimination index was determined for each item by using a table prepared by Flanagan (16, 472) which gives correlation coefficients between item scores and total test scores based on the upper and lower parts of the test score distribution. A summary of item data is given in Table 3. Decimal points were omitted in recording the indexes.

The Kuder-Richardson Formula 20 coefficient of reliability was computed from the data furnished by the upper 27 per cent and the lower 27 per cent of those students who took the final version of the test. A coefficient of .85 was obtained.

* The range of the difficulty index is from 1 to 100. The lower the index the more difficult the item.

** The range of the discrimination index is from -100 to 100. The higher the index the better the discrimination.

TABLE 3
ACHIEVEMENT TEST ITEM DATA

Part I			Part II			Part III		
Item	Difficulty Index*	Discrimination Index**	Item	Diff. Index	Disc. Index	Item	Diff. Index	Disc. Index
1	81	47	1	89	38	1	88	20
2	62	50	2	86	59	2	86	16
3	76	50	3	83	39	3	93	28
4	81	49	4	77	32	4	43	40
5	75	49	5	70	67	5	92	49
6	58	46	6	73	68	6	70	47
7	70	31	7	72	43	7	74	49
8	66	60	8	73	64	8	73	51
9	44	21	9	62	64	9	75	36
10	70	72	10	41	51	10	62	22
11	57	40	11	55	58	11	74	29
12	66	46	12	45	48	12	71	54
13	68	50	13	58	43	13	71	18
14	44	61	14	40	75	14	62	38
15	29	10	15	63	65	15	56	22
16	43	43	16	30	72	16	48	41
17	62	50	17	30	61	17	33	30
18	35	35	18	31	60	18	36	13
19	58	45	19	28	46	19	32	24
20	52	20	20	6	16	20	16	0

* The range of the difficulty index is from 1 to 100. The lower the index the more difficult the item.

** The range of the discrimination index is from -100 to 100. The higher the index the better the discrimination.

The coordinator also furnished intelligence test scores which were available in each school at that time. The decision to use existing intelligence test data rather than to give the same intelligence test was influenced by

CHAPTER III
PROCEDURE AND RESULTS

- (i) The administration of the STEP tests had already taken two class periods.
- Procedure

The study was conducted during the months of January, February, and March, 1964, with the cooperation of the teachers and administrators of the participating schools. A list of participating coordinators and teachers is given in Appendix C. During the summer of 1963, these coordinators had agreed to use the material with selected classes in their schools. The choice of classes was made by each coordinator after consultation with the author. In January, 1964, the mathematics section of the Sequential Tests of Education Progress (STEP) (6) was given to all potential subjects except those in Colorado. Students in grade 8 or grade 9 took Form 3B; students in grade 10, 11, or 12 took Form 2A.

The aim of the STEP mathematics test is to measure the extent to which the important objectives of mathematics in general education have been achieved. They concentrate on assessment of developed mathematical concepts, abilities, and skills considered essential for the mathematical literacy of the average, well-informed student. (7, 7)

The raw scores for all tests within an area, regardless of form, are converted to a single score scale. This makes it possible to compare a student's performance with that of students who have taken a different level of test and to trace an individual's development over a period of years. (7, 6)

Materials were mailed to the schools in January and followed by a personal visit by the author to the schools in Colorado, Nevada, and Oregon. During this visit the procedure to be followed in use of the books was discussed with the teachers. Arrangements were made with the New Jersey teachers by letter and telephone.

The coordinators also furnished intelligence test scores which were available in each school at that time. The decision to use existing intelligence test data rather than to ask all teachers to give the same intelligence test was influenced by the considerations given below:

- (1) The administration of the STEP tests had already taken two class periods.
- (2) Intelligence test data from different tests would increase the probability that a teacher could find information relevant to the students for whom he was considering the use of the text.

The intelligence tests which had been used are the California Test of Mental Maturity (CTMM), the Otis Gamma, the High School Placement Test of Scholastic Testing Service (HSPT-STS), the High School Placement Test of Science Research Associates (HSPT-SRA), the Differential Aptitude Test of the Psychological Corporation (DAT), and the Lorge-Thorndike Intelligence Test (LTIT).

Each coordinator furnished class lists on which available intelligence test scores were listed. For ease in recording subject data, identification numbers were assigned to subjects. The first digit in each identification number indicated the school, the second digit the grade level, '0' for grade 8, '1' for grade 9, '2' for grade 10, '3' for grade 11, and '4' for grade 12. The last two digits identified a particular student. For example, the identification number 1211 identified the subject as student number 11 in the tenth grade at Pascack Valley High School.

All materials were mailed to the schools in January and followed by a personal visit by the author to the schools in Colorado, Nevada, and Oregon. During this visit the procedure to be followed in use of the books was discussed with the teachers. Arrangements were made with the New Jersey teachers by letter and telephone.

Each student was told his identification number, received a copy of the programed text and a copy of the workbook. The identification number was used on the workbook and the test answer sheets. In some classes the teachers allotted class time for use of the books; in others the books were used only outside of class. Teachers were advised to give students no assistance except when the student was directed to his teacher by a note in the book (See page 239 of Rules of Reasoning). Each student was permitted to proceed at his own rate. However, the student was required to report periodically to his teacher on his progress. After all students in a class had completed the book, the final examination was given. The time from the beginning of the book to the taking of the test ranged from three to five weeks. Students who finished early were given reading assignments in books such as Flatland and The Education of T. C. Mils or used the time for remedial work related to their regular classroom assignments. The Barrington trial had indicated that 40 minutes was sufficient time for all students to complete the test. This enabled teachers to give the final test in one class period. The test answer sheets and the workbooks were returned to the author.

Results

The data on post-study performances were analyzed with the assistance of members of the UICSM Objectives Studies Department.

The achievement test was scored on the basis of one point for each item correct and no points for incorrect or omitted responses. An individual test score indicated the number of correct responses. No correction formula was used.

Data for subjects grouped by grade level are summarized in Table 4. Data from the subjects in Colorado were not used in this part of the analysis since these subjects had not taken the STEP test.

TABLE 4
MEANS AND STANDARD DEVIATIONS
OF ACHIEVEMENT TEST SCORES AND STEP TEST SCORES
GRADE LEVEL GROUPING

Grade	N	Logic Test		STEP Test	
		Mean	S. D.	Mean	S. D.
8	26	33.7	6.8	281.7	5.8
9	151	35.8	7.3	284.5	8.4
10	106	33.2	9.3	282.1	8.5
11	126	38.7	5.3	291.9	10.9
12	44	41.1	8.1	301.0	7.2

Table 4 indicates that the grade level of a student makes little difference in his achievement score after studying the programed logic text. A teacher who is considering the use of this text should consider the possible advantages in having a group of ninth graders familiar with the material presented in the text even though the data indicate that these students would do slightly better if the text were presented in the eleventh grade. The high mean for students in grade 12 is consistent with the high mean on the STEP test.

Data for subjects grouped by grade level and type of mathematics course are summarized in Table 5. The Colorado subjects were also omitted from this grouping.

TABLE 5
MEANS AND STANDARD DEVIATIONS
OF ACHIEVEMENT TEST SCORES AND STEP TEST SCORES
GRADE LEVEL AND MATHEMATICS COURSE GROUPING

Grade	Type of Mathematics Course	N	Logic Test		Percentile Band	STEP Test	
			Mean	S. D.		Mean	S. D.
8	UICSM	26	33.7	6.8	99-100	282	5.8
9	UICSM	117	37.9	6.2	91-98	286	6.6
9	Other	34	28.7	6.0	76-91	278	10.6
10	UICSM	35	42.2	7.1	80-94	286	9.3
10	Other	71	28.7	6.4	67-86	280	8.0
11	UICSM	48	37.5	5.8	62-86	285	9.1
11	Other	78	39.4	4.9	92-98	296	9.8
12	Other	44	41.1	8.1	94-99	301	7.2

Table 5 shows that students in grades 9 and 10 with a UICSM background have a higher mean score on the logic test than do those who have not studied UICSM mathematics. A subjective analysis of the UICSM texts for grades 9 and 10 and the texts studied by other students suggests that the UICSM students have had more practice in using deductive reasoning. The higher means on the STEP test for the UICSM students in grade 9 and 10 also suggest that these students began the study of the logic text with more knowledge of "mathematical concepts, abilities, and skills considered essential for the mathematical literacy of the average well-informed student" (7, 7).

Data for subjects grouped by grade level and type of intelligence test are summarized in Table 6. This information is provided for teachers who may want to use intelligence test scores as one criterion for selecting students to study the programed text.

Teachers are urged to use caution when applying the data given in Table 6. Since intelligence quotients obtained from different tests are generally not comparable, student data from, say, the California Test of Mental Maturity should not be used as a basis for comparing those students with students for whom data from the Otis Gamma were used. It is hoped that each teacher will be able to find in Table 6 data which are directly applicable to a particular student or class.

TABLE 6
MEANS OF ACHIEVEMENT TEST SCORES
AND INTELLIGENCE TEST SCORES

Grade	N	Type of Intelligence Test	Intelligence Test Mean	Logic Test Mean
8	26	CTMM	122.8	33.7
9	25	Otis Gamma	114.9	33.2
9	39	SPS-HSP	126.2	35.8
9	57	SRA-HSP	120.5	38.2
9	36	Large-Thorndike	122.0	40.3
10	31	Otis Gamma	107.4	29.7
10	27	STS-HSP	107.7	25.6
10	20	CTMM	120.7	46.5
11	74	Otis Gamma	117.7	39.5
11	47	DAT	85 *	37.4
12	40	STS-HSP	115.3	40.4

* The data for DAT test scores were given in percentiles. The median percentile rank for the 47 eleventh grades was 85.

CHAPTER IV

SUMMARY AND CONCLUSIONS

This study was designed to develop a programed text covering selected topics in formal logic and to collect data which indicated the effectiveness of the text for different groups of students. The selection of topics for the text was based on the relevance of these topics to the study of secondary school mathematics.

The sample consisted of students from grades 8, 9, 10, 11, and 12 in four school systems. All subjects studied a programed text based on the appendix to Unit 6, High School Mathematics. After completing the book, each subject was given a test designed to measure his understanding of the material. The test results for different groups of students were analyzed and reported for the information of teachers who are considering the use of the text.

Although each teacher must decide for himself if the book should be used with a particular student or group of students, some author comments appear to be in order. An analysis of the test results together with comments from teachers who used the book led to the following conclusions:

- (1) The text is suitable for mathematics students at any of the grade levels 9 through 12 and for students in grade 8 who have begun the study of algebra.
- (2) The achievement score of a student is affected by the type of mathematical background he has as well as by his knowledge of mathematical topics.
- (3) Students of all grade levels found the book challenging and interesting.

Conclusion (1) is supported by the data given in Table 4. Although there are differences in means between grade levels, the range of means (33.2 for tenth graders to 41.1 for twelfth graders) suggests to the author that the data do not indicate that any particular grade level is the place to have students study the logic text.

Support for conclusion (2) comes from the data summarized in Table 5. The STEP test mean for tenth grade students with UICSM background was 286; the STEP test mean for tenth grade students who had not studied the UICSM program was 280. The logic test mean for the UICSM group was 42.2; the logic test mean for the other group was 28.7. Similar differences were shown by students in grade 9. The difference in logic test means for eleventh grade students was small, (37.5 for UICSM students, 39.4 for those studying other programs) although the STEP test mean for UICSM students was 285 and the STEP test mean for other students was 296. These data for students in grades 9, 10, and 11 suggest to the author that a student's logic test score was affected by the type of mathematics course he had studied. A subjective analysis of the textbooks used by students indicated that the UICSM students would have had more practice in using deductive reasoning and would be more familiar with the language used in the logic text. Hence, a teacher who decides to use the logic text with students who have not studied the UICSM texts may wish to have them do supplementary work designed to provide them with additional practice in deductive reasoning. Such practice could also increase their familiarity with the language used in the text.

Support for conclusion (3) came from teacher comments written during the time students were studying the text. For example, one teacher wrote,

I think it can safely be said that the students (most of them anyway) worked diligently on the logic, and had fun doing it.

Teacher comments and student notes in the workbook suggest that increased learning may be brought about by using class discussion to supplement the study of the text. This suggests a study in which the programmed text is used together with teacher instruction. Such a study might uncover certain teaching techniques which could then be added to the existing programmed text to increase its usefulness as a text to be used without teacher assistance.

The major result of the present study is a text covering selected topics in formal logic and a test designed to measure understanding of these topics. A study which is designed to provide teachers with information about the correlation between a student's logic score and his I. Q. or some mathematics achievement score would be useful. Such information could be used by teachers who are considering the use of the logic text.

In general, the text was judged to be suitable for mathematics students at any of the grade levels 9 through 12. The text is also suitable for students in grade 8 who have begun the study of algebra. Of course, each teacher must decide for himself if the book should be used with a particular student or group of students. It is hoped that the information provided herein will be of help to the teacher in making this decision.

RULES OF REASONING

Universal Instantiation (UI)

Each instance of a universal generalization is a consequence of it.

Test Pattern Principle (TPP)

A universal generalization is a consequence of given sentences if there is a pattern for showing that any instance of the generalization is a consequence of the given sentences.

The Replacement Rule for Equations (RRE)

Given an equation and another sentence, if one side of the equation is replaced by the other side somewhere in the sentence, the new sentence thus obtained is a consequence of the given equation and sentence.

Law of Identity (LI)

The sentence ' $\forall x \ x = x$ ' is an acceptable premiss in a derivation.

Modus Ponens (MP)

$$\frac{p \quad \text{if } p \text{ then } q}{q}$$

is a valid inference pattern.

Conditionalizing (C)

$$\frac{q}{\text{if } p \text{ then } q}$$

is a valid inference pattern.

The Discharge Rule (D)

If the consequent of a conditional sentence is a consequence of the antecedent of the conditional sentence, and other premisses, then the conditional sentence is a consequence of the other premisses alone.

Double Denial and Reverse Double Denial

Given a statement P and its negation $\text{not } P$, if one side of the biconditional is replaced by the other side somewhere in the other, the resulting statement obtained is a consequence of the given statement.

$$\frac{P}{\text{not } [\text{not } (P)]} \quad \text{and} \quad \frac{\text{not } [\text{not } (P)]}{P}$$

are valid inference patterns.

Modus Tollens (MT)

$$\frac{\text{if } p \text{ then } q \quad \text{not } q}{\text{not } p}$$

is a valid inference pattern.

Contraposition

$$\frac{\text{if } p \text{ then } q}{\text{if not } q \text{ then not } p}$$

$$\frac{\text{if not } p \text{ then } q}{\text{if not } q \text{ then } p}$$

$$\frac{\text{if } p \text{ then not } q}{\text{if } q \text{ then not } p}$$

$$\frac{\text{if not } p \text{ then not } q}{\text{if } q \text{ then } p}$$

are valid inference patterns.

Conjunctions

$$\frac{p \quad q}{p \text{ and } q}$$

$$\frac{p \text{ and } q}{p}$$

$$\frac{p \text{ and } q}{q}$$

$$\frac{\text{not } (p \text{ and } q)}{\text{not } p \text{ or not } q}$$

are valid inference patterns.

Alternations

$$\frac{p}{p \text{ or } q}$$

$$\frac{q}{p \text{ or } q}$$

$$\frac{p \text{ or } q \quad \text{not } q}{p}$$

$$\frac{p \text{ or } q \quad \text{not } p}{q}$$

$$\frac{p \text{ or } q \quad \text{if } p \text{ then } r \quad \text{if } q \text{ then } r}{r}$$

are valid inference patterns.

The Replacement Rule for Biconditionals

Given a biconditional sentence and another sentence, if one side of the biconditional sentence is replaced by the other side somewhere in the other sentence, the new sentence thus obtained is a consequence of the given sentences.

Biconditionals

$$\frac{p \text{ if and only if } q}{(if \ p \ \text{then } q) \text{ and } (if \ q \ \text{then } p)} \qquad \frac{(if \ p \ \text{then } q) \text{ and } (if \ q \ \text{then } p)}{p \text{ if and only if } q}$$

are valid inference patterns.

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APPENDIX A
RULES OF REASONING

The text material used by the students differs in two ways from the copy given here.

In the student text, question marks and answers were printed in green ink. Green ink was used to help the student locate the answers to questions and to help him see places which required written responses.

A typical sheet of the student text contained printing on both sides. For example, page 86 was printed on one side of a sheet and page 193 was printed upside down on the other side of that sheet. This arrangement reduced the size of the text and at the same time kept the answers covered until a student had answered the questions. The student proceeded through the book studying "right-hand" pages through page 137. He was then instructed to turn the book around and read the facing page. He then continued through the book studying the pages which had originally appeared as upside down "left-hand" pages. At some places in the text, an "a-page" faced a "right-hand" page. For example, pages 75a and 75 are facing pages. This arrangement was used when a single page was not sufficient for the presentation of a topic or a discussion of the answers to exercises.

UICSM High School Mathematics

RULES OF REASONING

A Programed Edition

William T. Hale

University of Illinois Committee on School Mathematics

[RR]

HOW TO USE THIS BOOK AND WORKBOOK

Write your name, ID number (given to you by your teacher), and the name of your school on the cover of the Workbook. Do it now!

Write your answers in your workbook. At the top of each page of the book are given the correct answers for questions on the preceding page. After writing your answers on the work sheet, check them with the correct answers given in the book. Then record your results for that page. Use this method:

Put an ' X ' on any wrong answer, or anyplace you omitted an answer.

If you answered all the items on a page correctly, put a check mark (✓) by that page numeral on the work sheet.

[Your work sheet is not a test. You will be given a test after you have finished this book.]

Remember, work at the rate which is best for you.

In your previous work in mathematics you may have proved some theorems. In this book we shall discuss some of the rules of reasoning used in proving theorems.

The book is designed for you to use with a minimum of help from your teacher. Read each page carefully. If you get some answers wrong, try to see why you were wrong. There will often be explanatory material given with the answers. You may find it helpful to review some of the pages when you are having trouble. Sometimes the material on following pages may help you clear up your difficulties.

Work at the rate which is best for you .

[RR]

[Page 1]

Write, on your work sheet, the letters which correspond with true sentences.

(a) 5×3 is odd.

(b) 8×6 is even.

(c) 8×3 is odd.

(d) 5×6 is even.

(e) 31×3 is odd.

(f) 31×6 is even.

[This is just a practice exercise to show you how to record results. The correct letters have been written on your work sheet. See if you agree with the answers.]

[RR]

[Page 2]

Just the true sentences are printed in green.

(a) 5×3 is odd.(b) 8×6 is even.(c) 8×3 is odd.(d) 5×6 is even.(e) 31×3 is odd.(f) 31×6 is even.

Record your results. [This has already been done for this practice exercise. Since there were no mistakes, we put a check mark (✓) by 'Page 1' on the work sheet.]

* * *

Write, on your work sheet, the letters which correspond with true sentences.

(a) 7×6 is an even number.(b) 33×6 is an even number.(c) $4,362,871 \times 6$ is even.(d) $\{[(33 - 8) + 2] \div 9\} \times 6$ is even.

[This has already been done for you on your work sheet. A mistake has been made on purpose to show you how to record your results.]

[RR]

[Page 3]

The true sentences are printed in green.

(a) 7×6 is an even number.

(b) 32×6 is an even number.

(c) $4,362,871 \times 6$ is even.

(d) $\{[(33 - 8) + 2] \div 9\} \times 6$ is even.

The answer for (c) was wrong, so you should put an 'X' between 'b' and 'd' on the work sheet.

* * *

We are quite sure that you would have circled '(c)'. In fact, you probably know that

$$(8,241,036,861,763 - 2,362,492) \times 6$$

is an even number. [It is.] How about $9,263,831,367,752,011,201,515 \times 6$? Is it even? You probably said "yes" without doing any multiplication at all. You may even be thinking something like:

(1) Any number multiplied by 6 is even.

or:

(2) No matter what number you pick, the result of multiplying it by 6 is even.

(1) and (2) are universal statements in ordinary English. Here is how we say the same thing using more formal language:

For each number x , $x \cdot 6$ is even.

Here is another universal statement in ordinary English:

Whatever real number you pick, the result of multiplying it by 0 is 0.

Here is how we say the same thing using a universal generalization:

For each number y , $y \cdot 0 = 0$.

Translate, on your work sheet, the following sentence to a universal generalization.

The result of multiplying any number by 1 is that number.

[RR]

[Page 4]

For each number x , $x \cdot 1 = x$.

or:

For each number y , $y \cdot 1 = y$.

If you were correct, put a check mark by 'Page 3' on your work sheet. If you were wrong, put an 'X' on your answer.

* * *

Some mathematicians use ' \forall ' as an abbreviation for 'For each'. Since most of the statements in this book will be about numbers, we shall further abbreviate 'For each number x ' to ' \forall_x '. Hence, the universal generalization:

For each number x , $x \cdot 0 = 0$.

can be abbreviated to:

$$\forall_x x \cdot 0 = 0$$

Complete on your work sheet.

- (a) ' \forall_x ' is an abbreviation for: _____ ?
- (b) 'For each number y ' is abbreviated to: _____ ?
- (c) ' \forall_z ' is an abbreviation for: _____ ?
- (d) 'For each number w ' is abbreviated to: _____ ?

[RR]

[Page 5]

- (a) ' \forall_x ' is an abbreviation for: For each number x
- (b) 'For each number y ' is abbreviated to: \forall_y
- (c) ' \forall_z ' is an abbreviation for: For each number z
- (d) 'For each number w ' is abbreviated to: \forall_w

[Be sure to mark any wrong answers with an (X). If each of your answers was correct, put a check mark (✓) by 'Page 4'.]

* * *

Write the following universal generalization in abbreviated form by using ' \forall_y '.

- (a) For each number y , $y \cdot 1 = y$.

Now, write in unabbreviated form the universal generalization given below.

- (b) $\forall_z z \cdot 3 = 3 \cdot z$

[RR]

[Page 6]

$$(a) \quad \forall_y y \cdot 1 = y$$

$$(b) \quad \text{For each number } z, z \cdot 3 = 3 \cdot z$$

[Recall that ' \forall_z ' is an abbreviation for 'For each number z '.]

* * *

On your work sheet, write the letters which correspond with true sentences.

$$(a) \quad \text{For each number } x, x \cdot 6 \text{ is even.}$$

$$(b) \quad \forall_x x \cdot 0 = 0$$

$$(c) \quad \forall_y y \cdot 1 = y$$

$$(d) \quad \text{For each number } z, z \cdot 3 = 3 \cdot z.$$

[RR]

[Page 7]

Just the true sentences are printed in green.

(a) For each number x , $x \cdot 6$ is even.

(b) $\forall_x x \cdot 0 = 0$

(c) $\forall_y y \cdot 1 = y$

(d) For each number z , $z \cdot 3 = 3 \cdot z$.

* * *

If you thought that the sentence:

For each number x , $x \cdot 6$ is even

was true, you probably overlooked numbers like $\frac{1}{2}$ and $\frac{1}{6}$. $\frac{1}{2} \cdot 6 = 3$ and 3 is not even.

The universal generalization:

For each number x , $x \cdot 6$ is even

tells you that no matter what number you pick, the result of multiplying that number by 6 is even.

Maybe this is the universal statement you were thinking about:

- (1) No matter what whole number you pick, the result of multiplying it by 6 is even.

Here is an incomplete translation of (1):

For each number x , if x is a _____ ?
then $x \cdot 6$ is _____ ?

On your work sheet, write the complete translation of (1).

[RR]

[Page 8]

For each number x , if x is a whole number
then $x \cdot 6$ is even.

* * *

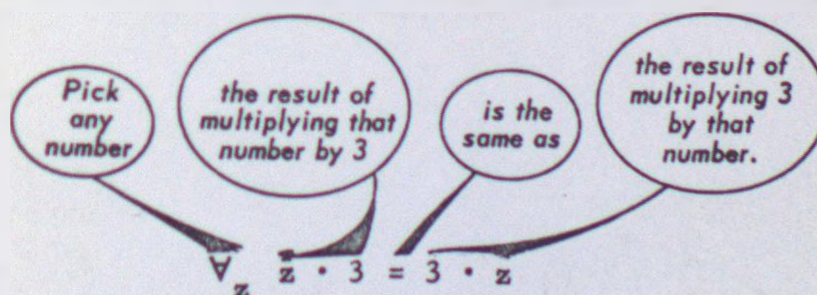
The universal generalization:

$$\forall_z z \cdot 3 = 3 \cdot z$$

tells you the same thing as the paragraph:

Pick any number. The result of multiplying
that number by 3 is the same as the result of
multiplying 3 by that number.

The corresponding parts are shown here.



On your work sheet write a universal generalization which is a translation
of the paragraph given below. Use ' \forall_x '.

Pick any number. The result of adding 5 to
that number is the same as the result of
adding that number to 5.

[RR]

[Page 9]

$$\forall_x x + 5 = 5 + x$$

* * *

Translate each of the paragraphs given below.

- (a) Pick any number. The result of adding the number to itself is the same as the result of multiplying 2 by the number.
- (b) No matter what number you pick, the result of multiplying it by $\frac{1}{2}$ is the same as the result of dividing the number by 2.

[RR]

[Page 10]

$$(a) \quad \forall_x \quad x + x = 2 \cdot x$$

$$(b) \quad \forall_y \quad y \cdot \frac{1}{2} = y \div 2 \quad \text{or:} \quad \forall_y \quad y \cdot \frac{1}{2} = \frac{y}{2}$$

[You are also correct if you used 'y' or 'z' when we used 'x' or if you used 'x' or 'z' where we used 'y'.]

* * *

We have said that the universal generalization:

$$\forall_z \quad z \cdot 3 = 3 \cdot z$$

is true.

Write on your work sheet, the letters which correspond with true generalizations.

$$(a) \quad \forall_z \quad z(3.14159) = (3.14159)z$$

$$(b) \quad \forall_z \quad z(-17) = (-17)z$$

$$(c) \quad \forall_z \quad z\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)z$$

$$(d) \quad \forall_z \quad z(\pi - 13) = (\pi - 13)z$$

$$(e) \quad \forall_z \quad z(23) = (2)z$$

$$(f) \quad \forall_z \quad z(35) = (35)z$$

[RR]

[Page 11]

The true sentences are printed in green.

$$(a) \forall_z z(3.14159) = (3.14159)z$$

$$(b) \forall_z z(-17) = (-17)z$$

$$(c) \forall_z z(\frac{1}{2}) = (\frac{1}{2})z$$

$$(d) \forall_z z(\pi - 13) = (\pi - 13)z$$

$$(e) \forall_z z(23) = (2)z$$

$$(f) \forall_z z(35) = (35)z$$

* * *

The universal generalization:

$$\forall_z z(3.14159) = (3.14159)z$$

says the same thing as the universal statement:

No matter what number you pick, the result of multiplying it by 3.14159 is the same as the result of multiplying 3.14159 by it.

Pick any number. The universal generalization:

$$\forall_y y(-17) = (-17)y$$

tells you that the result of multiplying it by -17 is the same as the result of multiplying -17 by it .

Suppose you multiply your number by 83 and I multiply 83 by your number . Will we get the same result, provided our computing is correct? The answer is yes.

Translate to a universal generalization:

No matter what number you pick, the result of multiplying your number by 83 is the same as the result of multiplying 83 by your number.

[RR]

[Page 12]

$$\forall x \quad x \cdot 83 = 83 \cdot x$$

* * *

You are probably wondering why we picked -17 and 83 to use in our discussion. We might just as well have picked 98 or 8,365 or even -98.5. In fact, you probably believe that the paragraph given below is true.

- (1) Pick any number. Now, pick a second number. Multiply the first by the second. Now, multiply the second number you picked by the first. The products are the same.

We have seen how to translate a phrase like 'Pick any number' into our formal language. So we can start to translate the paragraph (1) into a universal generalization. We have seen that the way to translate 'Pick any number' is 'For each number x '. How shall we translate 'Now, pick a second number' into our formal language? 'Pick a number' suggests 'For each number x ', but in order to translate 'second number' we need to use a letter different from ' x '.

Consider this translation:

- (2) For each number x , for each number y , $xy = yx$

The paragraph (1) and the universal generalization (2) tell you the same thing. In this book we will emphasize the universal generalization form. You may occasionally need to translate to ordinary English.

Translate, on your work sheet, the following paragraph into a universal generalization.

Pick any number. Now pick a second number.
Add the second number to the first number.
The result is the same as the result of adding
the first number to the second number.

[RR]

[Page 13]

$$\forall_x \forall_y x + y = y + x$$

or:

For each number x , for each number y , $x + y = y + x$

* * *

- (a) Write, on your work sheet, the following universal generalization in abbreviated form by using ' \forall_y '.

For each number y , $y \cdot 1 = y$.

- (b) Now, write in unabbreviated form [as (a) is written above] the universal generalization given below.

$$\forall_x \forall_y y > x$$

[RR]

[Page 14]

(a) $\forall_y y \cdot 1 = y$

(b) For each number x , for each number y , $y > x$.

[The generalization ' $\forall_x \forall_y y > x$ ' is false, but we won't worry about that now!]

* * *

(a) Write, on your work sheet, the following universal generalization in abbreviated form by using ' \forall_z '.

For each number z , $z + 2 = 7$.

(b) Write in unabbreviated form [as (b) is written above] this universal generalization:

$$\forall_y \forall_x y + x = x + y$$

[RR]

[Page 15]

(a) $\forall_z z + 2 = 7$

(b) For each number y , for each number x , $y + x = x + y$

* * *

Now, we know that it is **not** the case that no matter what number we pick, the result of adding 2 to that number is 7.

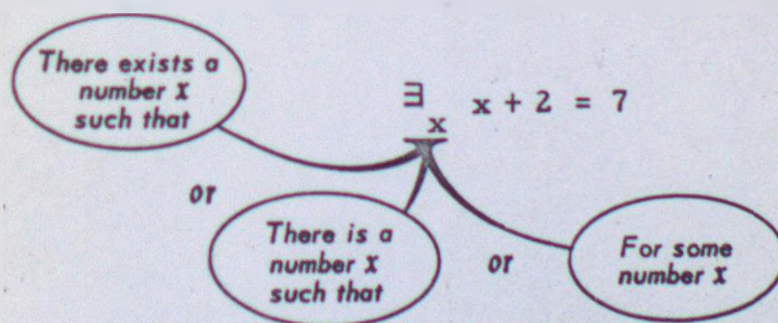
However, it is the case that

- (1) there is some number such that when 2 is added to the number the result is 7.

(1) can be abbreviated to:

There is a number x such that $x + 2 = 7$.

In our formal language we write the **existential generalization**:



On your work sheet, complete this sentence:

The existential generalization:

$$\exists_y y - 3 = 9$$

can be unabbreviated to:

$$\underline{\quad ? \quad} y - 3 = 9$$

[RR]

[Page 16]

The existential generalization:

$$\exists_y y - 3 = 9$$

can be unabbreviated to:

There exists a number y such that $y - 3 = 9$

or:

There is a number y such that $y - 3 = 9$

or:

For some number y , $y - 3 = 9$

* * *

On your work sheet, abbreviate the existential generalizations given below.

(a) There is a number z such that $z \cdot 3 = 8$.

(b) There exists a number x such that $x^2 \neq 2$.

(c) For some number y , $y > 3$.

[RR]

[Page 17]

(a) $\exists_z z \cdot 3 = 8$

(b) $\exists_x x^2 = 2$

(c) $\exists_y y > 3$

* * *

Note that the existential generalization ' $\exists_z z \cdot 3 = 8$ ' is true since there is a number, namely $\frac{8}{3}$, such that the product of that number, $\frac{8}{3}$, by 3 is 8. The existential generalization ' $\exists_y y > 3$ ' is also true since there is a number, say, 5, such that that number, 5, is greater than 3. Of course, 5 is not the only number greater than 3, but all the existential generalization asserts is that there is a number greater than 3. There is at least one number greater than 3. Since 5 is such a number, the generalization is true.

The universal generalization:

$$\forall_x \exists_y y > x$$

can be unabbreviated to:

For each number x , there is a number y such that $y > x$.

Write, on your work sheet, in unabbreviated form the generalization:

$$\forall_x \exists_y x \cdot y = x$$

[RR]

[Page 18]


For each number x , there is a number y such that $x \cdot y = x$

or: For each number x , there exists a number y such that $x \cdot y = x$

or: For each number x , for some number y , $x \cdot y = x$

* * *


The sentence:



$$\forall x \exists y x \cdot y = x$$

is a universal generalization.

The sentence



$$\exists y \forall x x \cdot y = x$$

is an existential generalization.

Here are some sentences. On your work sheet, write just those letters which correspond to generalizations (universal or existential).

(a) $\forall x x \cdot 0 = 0$

(b) $\forall w w + 1 = w$

(c) $\forall y \forall x yx = xy$

(d) $a + b = b + a$

(e) $5 + 2 = 7$

(f) $\exists x x + 2 = 7$

(g) $\exists z \forall x x \cdot z = z$

(h) $\forall x x + 2 = 7$

(i) $\forall x \forall y x + y > x$

(j) $\forall x \exists y y > x$

[RR]

[Page 19]

The generalizations are printed in green.

$$(a) \forall_x x \cdot 0 = 0$$

$$(b) \forall_w w + 1 = w$$

$$(c) \forall_y \forall_x yx = xy$$

$$(d) a + b = b + a$$

$$(e) 5 + 2 = 7$$

$$(f) \exists_x x + 2 = 7$$

$$(g) \exists_z \forall_x x \cdot z = z$$

$$(h) \forall_x x + 2 = 7$$

$$(i) \forall_x \forall_y x + y > x$$

$$(j) \forall_x \exists_y y > x$$

[(f) and (g) are existential generalizations. (d) is an open sentence.]

* * *

Which of these are universal generalizations? On your work sheet write the appropriate letters.

$$(a) \forall_x x \cdot 0 = 0$$

$$(b) \forall_y y + 1 = y$$

$$(c) a + b = b + a$$

$$(d) \exists_x x + 2 = 7$$

$$(e) \exists_z \forall_x xz = z$$

$$(f) \forall_x \exists_y y > x$$

[RR]

[Page 20]

The universal generalizations are printed in green.

$$(a) \forall_x x \cdot 0 = 0$$

$$(b) \forall_y y + 1 = y$$

$$(c) a + b = b + a$$

$$(d) \exists_x x + 2 = 7$$

$$(e) \exists_z \forall_x xz = z$$

$$(f) \forall_x \exists_y y > x$$

* * *

Here is a universal statement in ordinary English:

All numbers are odd.

Here is how we say the same thing using a universal generalization:

$$\forall_x x \text{ is odd}$$

Here is an existential statement in ordinary English:

Some number is greater than 0.

Here is how we say the same thing using an existential generalization:

$$\exists_x x > 0$$

Use the symbols ' \forall ', ' \exists ', ' x ', ' y ' and ' z ' and the standard signs of arithmetic and write these sentences as generalizations.

- (a) There is a number x greater than 7.
- (b) Given any number z there is a smaller number y .
- (c) For any two numbers x and y the difference of x from y is less than the difference of y from x .
- (d) Every number is greater than 0.

[RR]

[Page 21]

(a) $\exists_x x > 7$

(b) $\forall_z \exists_y y < z$

(c) $\forall_x \forall_y y - x < x - y$

(d) $\forall_x x > 0$ or: $\forall_y y > 0$ or: $\forall_z z > 0$

* * *

We are considering two kinds of generalization sentences, universal generalizations and existential generalizations.

For each incomplete sentence given below, write, on your work sheet, the word 'universal' or the word 'existential' to correctly complete the sentence.

(a) ' $\forall_x x \cdot 1 = x$ ' is a(n) _____ ? _____ generalization.

(b) ' $\exists_x x + 2 = 5$ ' is a(n) _____ ? _____ generalization.

(c) ' $\exists_x \forall_y y > x$ ' is a(n) _____ ? _____ generalization.

(d) ' $\forall_x \exists_y y > x$ ' is a(n) _____ ? _____ generalization.

[RR]

[Page 22]

- (a) ' $\forall_x x \cdot 1 = x$ ' is a(n) universal generalization.
- (b) ' $\exists_x x + 2 = 5$ ' is a(n) existential generalization.
- (c) ' $\exists_x \forall_y y > x$ ' is a(n) existential generalization.
- (d) ' $\forall_x \exists_y y > x$ ' is a(n) universal generalization.

* * *

Phrases such as 'For each number x ' [or its abbreviation ' \forall_x '] and 'There exists a number x such that' [or its abbreviation ' \exists_x '] are called quantifying phrases.

On your work sheet, write the quantifying phrase which would be used to convert sentences (a) and (b) to existential generalizations and the quantifying phrase which would be used to convert sentences (c) and (d) to universal generalizations.

(a) _____ ? _____ $x - 7 = 2$

(b) _____ ? _____ $2y = y^2$

(c) _____ ? _____ $x + 3 = 3 + x$

(d) _____ ? _____ $y + 2 = 9$

[RR]

[Page 23]

- | | | |
|-------------------------------------|---|-----------------------------|
| (a) $\exists_x \quad x - 7 = 2$ | } | Existential generalizations |
| (b) $\exists_y \quad 2y = y^2$ | | |
| (c) $\forall_x \quad x + 3 = 3 + x$ | } | Universal generalizations |
| (d) $\forall_y \quad y + 2 = 9$ | | |

* * *

On your work sheet write the letters which correspond with **universal generalizations**.

(a) $\forall_x \quad 1 + 2x = 3x$

(b) $\forall_x \quad x(2x + 3) = 2xx + 3x$

(c) $\forall_w \quad w \cdot 0 = w$

(d) $xy = yx$

(e) $\exists_y \forall_z \quad z \cdot y = 0$

(f) $\forall_x \forall_y \quad x + y > x$

{RR}

[Page 24]

Just the universal generalizations are printed in green.

(a) $\forall_x 1 + 2x = 3x$

(b) $\forall_x x(2x + 3) = 2xx + 3x$

(c) $\forall_w w \cdot 0 = w$

(d) $xy = yx$

(e) $\exists_y \forall_z z \cdot y = 0$

(f) $\forall_x \forall_y x + y > x$

* * *

Now, decide which of these sentences are true. Indicate your decision by writing on your work sheet the letters which correspond with true sentences.

(a) $\forall_x 1 + 2x = 3x$

(b) $\forall_x x(2x + 3) = 2xx + 3x$

(c) $\forall_w w \cdot 0 = w$

(d) $xy = yx$

(e) $\exists_y \forall_z zy = 0$

(f) $\forall_x \forall_y x + y > x$

[RR]

[Page 25]

(a) $\forall_x 1 + 2x = 3x$

True (b) $\forall_x x(2x + 3) = 2xx + 3x$

(c) $\forall_w w \cdot 0 = w$

(d) $xy = yx$

True (e) $\exists_y \forall_z zy = 0$

(f) $\forall_x \forall_y x + y > x$

[' $xy = yx$ ' is an open sentence and, hence, is neither true nor false.]

* * *

Which of these generalizations are true? Write the appropriate letters on your work sheet.

(a) $\forall_x x \cdot 0 = 0$

(b) $\forall_y y + 1 = y$

(c) $\forall_x \forall_y xy = yx$

(d) $\exists_x x + 2 = 7$

(e) $\exists_z \forall_x xz = z$

(f) $\forall_x x + 2 = 7$

(g) $\forall_x \forall_y x + y > x$

(h) $\exists_x x \cdot 1 = x$

(i) $\forall_y y > 0$

(j) $\forall_x x \cdot 1 = x$

[RR]

[Page 26]

True (a) $\forall_x x \cdot 0 = 0$

(b) $\forall_y y + 1 = y$

True (c) $\forall_x \forall_y xy = yx$

True (d) $\exists_x x + 2 = 7$

True (e) $\exists_z \forall_x xz = z$

(f) $\forall_x x + 2 = 7$

(g) $\forall_x \forall_y x + y > x$

True (h) $\exists_x x \cdot 1 = x$

(i) $\forall_y y > 0$

True (j) $\forall_x x \cdot 1 = x$

[Since there are many numbers x such that $x \cdot 1 = x$, there is certainly some number x such that $x \cdot 1 = x$. Hence, (h) is true.]

* * *

Now, on your work sheet, write the letters which correspond with universal generalizations.

(a) $\forall_x x \cdot 0 = 0$

(b) $\forall_y y + 1 = y$

(c) $\forall_y \forall_z zy = yz$

(d) $\exists_x x + 2 = 7$

(e) $\exists_z \forall_x xz = z$

(f) $\forall_x x + 2 = 7$

(g) $\forall_x \forall_y x + y > x$

(h) $\exists_x x \cdot 1 = x$

(i) $\forall_y y > 0$

(j) $\forall_x x \cdot 1 = x$

[RR]

[Page 27]

The universal generalizations are printed in green.

True (a) $\forall_x x \cdot 0 = 0$

(b) $\forall_y y + 1 = y$

True (c) $\forall_y \forall_z zy = yz$

True (d) $\exists_x x + 2 = 7$

True (e) $\exists_z \forall_x xz = z$

(f) $\forall_x x + 2 = 7$

(g) $\forall_x \forall_y x + y > x$

True (h) $\exists_x x \cdot 1 = x$

(i) $\forall_y y > 0$

True (j) $\forall_x x \cdot 1 = x$

* * *

Note that some of the universal generalizations, namely (a), (c), and (j), are true and others, namely (b), (f), (g), and (i), are not true.

Choose the correct word for each sentence, and write the complete sentence on your work sheet.

(a) A universal generalization ? be false.
(can/cannot)

(b) A false statement ? be a universal generalization.
(can/cannot)

(c) A true statement ? be a universal generalization.
(can/cannot)

(d) A universal generalization ? be true.
(can/cannot)

[RR]

[Page 28]

- (a) A universal generalization can be false.
- (b) A false statement can be a universal generalization.
- (c) A true statement can be a universal generalization.
- (d) A universal generalization can be true.

* * *

Since both true statements and false statements can be universal generalizations, you cannot use a "true-or-false" test to decide whether a given statement is a universal generalization. In fact, in several systems of formal logic the truth or falsity of a statement is of no interest. If we cannot use a "true-or-false" test for universal generalizations how then can we decide whether a given statement is a universal generalization? The use of the word 'formal' may give you a clue. A statement is a universal generalization if it is of a certain form. From the preceding exercises you have probably decided that

any statement which begins with
 ? or ? is a universal generalization,
 and
 if a statement is a universal generalization
 then the statement begins with ? or ? .

Write the complete sentence on your work sheet.

[RR]

[Page 29]

Any statement which begins with '∀' or 'For each'
is a universal generalization,

and

if a statement is a universal generalization then the
sentence begins with '∀' or 'For each'.

* * *

Write on your work sheet the letters which correspond with universal
generalizations.

(a) $\forall_x x \cdot 0 = 0$

(b) $\forall_z z + 1 = z$

(c) For each number x not equal to 0, $x \cdot \frac{1}{x} = 1$.

(d) For each set A , $A \cup \emptyset = A$.

[RR]

[Page 30]

The universal generalizations are printed in green.

$$(a) \quad \forall_x \quad x \cdot 0 = 0$$

$$(b) \quad \forall_z \quad z + 1 = z$$

$$(c) \quad \text{For each number } x \text{ not equal to } 0, \quad x \cdot \frac{1}{x} = 1.$$

$$(d) \quad \text{For each set } A, \quad A \cup \emptyset = A.$$

[You should have recognized that each of the sentences (a)-(d) is a universal generalization even if you do not understand what each one means. Each sentence has the proper form.]

* * *

Many mathematicians and logicians use the term 'universal generalization' when referring to sentences in ordinary English such as:

(a) Any number multiplied by 6 is even

(b) All men are mortal

(c) Each positive number is greater than 0

as well as when referring to the translations of (a), (b), and (c) into our formal language:

(a') $\forall_x \quad x \cdot 6 \text{ is even}$

(b') For each man x , x is mortal

(c') For each number x , if x is positive then $x > 0$

However, in this book we will use the term 'universal statement' when referring to sentences like (a), (b), and (c) and reserve the term 'universal generalization' for sentences like (a'), (b'), and (c') in our formal language.

Translate the universal statement:

All positive numbers are greater than 0

into our formal language. Use ' \forall_x '.

[RR]

[Page 31]

\forall_x if x is positive then $x > 0$.

* * *

Translate these sentences:

- (a) No matter what number you pick, the result of multiplying that number by 0 is 0.
- (b) Each even number is not prime.

into sentences in our formal language. Use ' \forall_y '.

[RR]

[Page 32]

$$(a) \quad \forall_y y \cdot 0 = 0$$

(b) \forall_y if y is even then y is not prime

* * *

It is not always clear whether a statement in ordinary English is a universal statement or an existential statement [a statement about the existence of something]. Consider the universal statement:

(A) All Freshmen are not smart.

Some people interpret (A) to mean

(B) No Freshmen are smart

or (C) There does not exist a smart Freshman

while others interpret (A) to mean

(D) Not all Freshmen are smart

or (E) There exists at least one Freshman
who is not smart

We consider (B) or (C) as the correct interpretation of (A), that is, both (B) and (C) are universal statements.

On your work sheet, write a sentence beginning with 'There does' which says the same thing as the sentence:

All dorses are not tall.

[RR]

[Page 33]

There does not exist a tall dorse.

* * *

Write on your work sheet the letters which correspond with universal statements.

- (a) There is a number greater than 100, 000, 000.
- (b) Each number is such that if it is even then it is not prime.
- (c) All numbers are prime.
- (d) There does not exist an odd perfect number.

[RR]

[Page 34]

Universal statements are printed in green.

- (a) There is a number greater than 100, 000, 000.
- (b) Each number is such that if it is even then it is not prime.
- (c) All numbers are prime.
- (d) There does not exist an odd perfect number.

* * *

Translate each of the sentences (a)-(d) above into a sentence in our formal language. We'll give you a start; you finish the job on your work sheet.

(a') $\exists_y y$ _____ ? _____

(b') For each number x , if x is even then _____ ? _____.

(c') For each number x , x is _____ ? _____.

(d') For each number y , if y is odd then _____ ? _____.

[RR]

[Page 35]

- (a) $\exists_y y > 100,000,000$
- (b) For each number x , if x is even then x is not prime.
- (c) For each number x , x is prime.
- (d) For each number y , if y is odd then y is not perfect.

* * *

The universal generalization:

$$\forall_x x \cdot 1 = x$$

is equivalent to the universal statement:

Whatever number you pick, the result of multiplying the number by 1 is that number.

Translate each of these into ordinary English.

(a) $\forall_x x + 0 = x$

(b) $\forall_x \forall_y x - y = x + -y$

[RR]

[Page 36]

- (a) Whatever number you pick, the result of adding 0 to that number is the number itself.
- (b) Pick a number. Now, pick a second number. The result of subtracting the second number from the first is the same as the result of adding the opposite of the second number to the first.

[For each generalization, we have given a translation to ordinary English. You may have written a different translation. If you think your translations are also correct, put a check mark on the work sheet.]

* * *

Translate to a universal generalization:

Pick a number. Pick a second number. Add the second number to the first number. Now pick a third number and add it to the sum of the first and second numbers. The result is the same as the result of adding the third number to the second number and adding that sum to the first number. [Whew!]

[RR]

[Page 37]

$$\forall_x \forall_y \forall_z (x + y) + z = x + (y + z)$$

* * *

With which would you rather work, the sentence from ordinary English or the universal generalization in our formal language? We prefer the universal generalization. You may sometimes find it helpful to translate from universal generalizations to ordinary English but we hope that you will become very familiar with the formal language and feel comfortable using it.

On the next page is a quiz. Write the answers for this quiz on your work sheet for Page 38.

QUIZ

A. Write, on your work sheet, the letters which correspond with sentences which are **universal sentences**.

- (a) No matter what real number you pick, the result of multiplying that number by 0 is 0.
- (b) Pick a number. Now, add the opposite of that number to that number. The result is 0.
- (c) There is some number such that 5 multiplied by that number is 10.
- (d) Pick a number. Now pick a second number. The result of subtracting the second number from the first is the same as the result of subtracting the first number from the second.
- (e) No number is prime.
- (f) Not all numbers are prime.

B. Translate (a) - (f), above, to universal generalizations or existential generalizations. Write the translations on your work sheet.

C. Write, on your work sheet, the letters which correspond with sentences which are **universal generalizations**.

(a) $\forall_z z + 0 = z$

(b) $\forall_x x + 2 = 5$

(c) $\exists_x \forall_y x + y = 0$

(d) $\forall_x \exists_y x + y = 0$

(e) $x + y = y + x$

(f) $\exists_x x + 2 = 5$

(g) $\forall_x \forall_y x + y > x$

(h) $\forall_x (x^2)^3 = x^6$

After writing your answers on your work sheet, turn this page over for the correct answers.

[Page 39a]

[RR]

Answers for quiz.

A. a b d e

B. (a) $\forall_x x \cdot 0 = 0$

(b) $\forall_x x + -x = 0$

(c) $\exists_x 5x = 10$

(d) $\forall_x \forall_y x - y = y - x$

(e) $\forall_x x$ is not prime

(f) $\exists_x x$ is not prime

C. a b d g h

If you had any wrong answers, review the first 37 pages and then go on to Page 39.

If all your answers were correct, go on to Page 39.

[RR]

[Page 39]

You have probably discovered many shortcuts while working in mathematics. One of the first things many people notice is that they get the same answer when they add, say, 2 to 7, as when they add 7 to 2. They soon realize that this is true of other numbers, not just 2 and 7. For example, when asked to add 97 to 3, they add 3 to 97 and are sure that the result is the same as if they had added 97 to 3. They also accept sentences like:

$$86 + 324 = 324 + 86$$

$$-38 + 14 = 14 + -38$$

without simplifying each side. Much later in their mathematical career this discovery is formally stated as the commutative principle for addition:

$$\forall_x \forall_y \quad x + y = y + x$$

Write the commutative principle for multiplication in concise form by using ' \forall '.

[RR]

[Page 40]

$$\forall_x \forall_y x \cdot y = y \cdot x$$

* * *

Suppose you were asked to write in concise form all the discoveries you have made about the real numbers and operations with the real numbers and all the things you have been taught about the real numbers and operations with them. To fulfill this request would take more paper and time than I could spare, how about you? However, we have seen that one way to indicate a great number of things like:

$$3 + 97 = 97 + 3$$

$$87 + 324 = 324 + 87$$

$$-38 + 14 = 14 + -38$$

$$5 + 2 = 2 + 5$$

$$-7 + 19 = 19 + -7$$

$$34.5 + \sqrt{2} = \sqrt{2} + 34.5$$

is to write:

$$\forall_x \forall_y x + y = y + x$$

We can get a good start on fulfilling the request for all we know about real numbers by writing a few basic principles.

On your work sheet for page 40, write in concise form the basic principles given there. [Some of these have been done for you. Some have been only started for you. The rest you must do completely by yourself.]

- (a) Commutative principle for addition [cpa]

$$\forall_x \forall_y x + y = y + x$$

- (b) Commutative principle for multiplication [cpm]

$$\forall_x \forall_y x \cdot y = y \cdot x$$

- (c) Associative principle for addition [apa]

$$\forall_x \forall_y \forall_z (x + y) + z = x + (y + z)$$

- (d) Associative principle for multiplication [apm]

$$\forall_x \forall_y \forall_z (x \cdot y) \cdot z = x \cdot (y \cdot z)$$

- (e) Distributive principle for multiplication over addition [dpma]

$$\forall_x \forall_y \forall_z (x + y) \cdot z = x \cdot z + y \cdot z$$

- (f) Distributive principle for multiplication over subtraction [dpms]

$$\forall_x \forall_y \forall_z (x - y) \cdot z = x \cdot z - y \cdot z$$

- (g) Principle for adding 0 [pa0]

$$\forall_x x + 0 = x$$

- (h) Principle for multiplying by 0 [pm0]

$$\forall_x x \cdot 0 = 0$$

- (i) Principle for multiplying by 1 [pm1]

$$\forall_x x \cdot 1 = x$$

- (j) Principle of opposites [po]

$$\forall_x x + -x = 0$$

- (k) Principle for subtraction [ps]

$$\forall_x \forall_y x - y = x + -y$$

[RR]

[Page 41]

See the facing page [41a] for answers.

* * *

The principles given on the facing page summarize many of the facts of arithmetic that we already know. Hence, we will take these principles as true sentences about arithmetic. There are many things which follow from these principles, that is, many things which are logical consequences of these principles. One of the rules of the game is that when we accept a principle we must accept the logical consequences of the principle.

One of the consequences of:

$$\begin{array}{ccc} \forall x & x + 0 = x & \\ | & & | \\ \downarrow & & \downarrow \\ & 5 + 0 = 5 & \end{array}$$

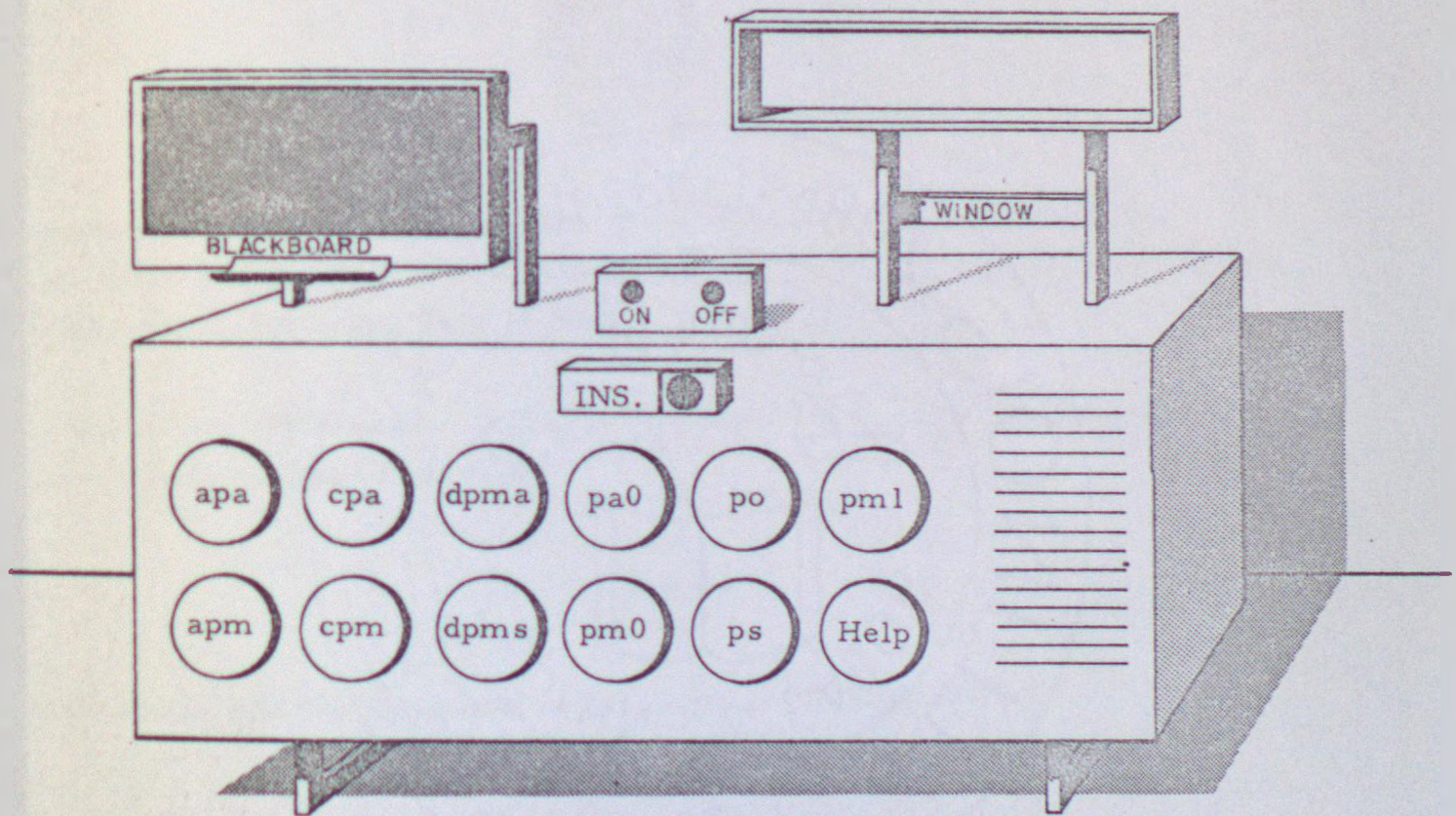
is, for instance:

Here is the instance of ' $\forall x \ x + 0 = x$ ' which corresponds with '-17':

$$-17 + 0 = -17$$

Write, on your work sheet,

- (a) the instance of ' $\forall x \ x + 0 = x$ ' which corresponds with '33.5',
- (b) the instance of ' $\forall x \ x + 0 = x$ ' which corresponds with '(38 + 3)',
- (c) the instance of ' $\forall x \ x + 0 = x$ ' which corresponds with '-7'.



[RR]

[Page 42]

Each of these is an instance of ' $\forall_x x + 0 = x$ '.

(a) $33.5 + 0 = 33.5$

(b) $(38 + 3) + 0 = (38 + 3)$

(c) $-7 + 0 = -7$

* * *

Writing an instance of a universal generalization requires very little thought. In fact, it is a mechanical procedure that a machine could perform. Pictured on the facing page is JIM, a Jiffy Instance Maker.

Here is how JIM works. In JIM's "memory" are stored the principles. If you push the button marked:

cpm

JIM shows you the commutative principle for multiplication

$$\forall_x \forall_y x \cdot y = y \cdot x$$

through the window.

Write, on your work sheet, what you would see through the window after you pushed the button marked:

apa

[RR]

[Page 43]

The associative principle for addition

apa

$$\forall_x \forall_y \forall_z (x + y) + z = x + (y + z)$$

* * *

On your work sheet write what you would see in the window if you pushed

(a)

cpm

(b)

dpma

(c)

po

(d)

pa0

(e)

apm

(f)

ps

(g)

apa

(h)

pm1

(i)

dpms

(j)

pm0

The chart at the end of this book may help you answer these questions and others about the principles.

[RR]

[Page 44]

- (a) cpm The commutative principle for multiplication

$$\forall_x \forall_y x \cdot y = y \cdot x$$
- (b) dpma The distributive principle for multiplication over addition

$$\forall_x \forall_y \forall_z (x + y) \cdot z = x \cdot z + y \cdot z$$
- (c) po The principle of opposites

$$\forall_x x + -x = 0$$
- (d) pa0 The principle for adding 0

$$\forall_x x + 0 = x$$
- (e) apm The associative principle for multiplication

$$\forall_x \forall_y \forall_z (x \cdot y) \cdot z = x \cdot (y \cdot z)$$
- (f) ps The principle for subtraction

$$\forall_x \forall_y x - y = x + -y$$
- (g) apa The associative principle for addition

$$\forall_x \forall_y \forall_z (x + y) + z = x + (y + z)$$
- (h) pm1 The principle for multiplying by 1

$$\forall_x x \cdot 1 = x$$
- (i) dpms The distributive principle for multiplication over subtraction

$$\forall_x \forall_y \forall_z (x - y) \cdot z = x \cdot z - y \cdot z$$
- (j) pm0 The principle for multiplying by 0

$$\forall_x x \cdot 0 = 0$$

* * *

Suppose JIM's window looked like this:

$\forall_x \forall_y x + y = y + x$

Which button was pushed? [Write your response on the work sheet.]

[RR]

[Page 45]

cpa

$$\forall_x \forall_y x + y = y + x$$

* * *

Answer the following on your work sheet.

	Button pushed	JIM's reply in window
(a)	<input type="radio"/>	$\forall_x \forall_y x \cdot y = y \cdot x$
(b)	<input type="radio"/>	$\forall_x x + 0 = x$
(c)	<input type="radio"/>	$\forall_x \forall_y x + y = y + x$
(d)	<input type="radio"/>	$\forall_x \forall_y \forall_z (x + y) + z = x + (y + z)$
(e)	<input type="radio"/>	$\forall_x \forall_y \forall_z (x \cdot y) \cdot z = x \cdot (y \cdot z)$
(f)	<input type="radio"/>	$\forall_x x + 0 = x$
(g)	<input type="radio"/>	$\forall_x \forall_y \forall_z (x - y) \cdot z = x \cdot z - y \cdot z$
(h)	<input type="radio"/>	$\forall_x x \cdot 0 = 0$
(i)	<input type="radio"/>	$\forall_x x + -x = 0$
(j)	<input type="radio"/>	$\forall_x \forall_y x + y = y + x$
(k)	<input type="radio"/>	$\forall_x \forall_y x - y = x + -y$
(l)	<input type="radio"/>	$\forall_x x \cdot 1 = x$
(m)	<input type="radio"/>	$\forall_x \forall_y \forall_z (x + y) \cdot z = x \cdot z + y \cdot z$

[RR]

[Page 46]

	Button pushed	JIM's reply in window
(a)	cpm	$\forall_x \forall_y x \cdot y = y \cdot x$
(b)	pa0	$\forall_x x + 0 = x$
(c)	cpa	$\forall_x \forall_y x + y = y + x$
(d)	apa	$\forall_x \forall_y \forall_z (x + y) + z = x + (y + z)$
(e)	apm	$\forall_x \forall_y \forall_z (x \cdot y) \cdot z = x \cdot (y \cdot z)$
(f)	pa0	$\forall_x x + 0 = x$
(g)	dpms	$\forall_x \forall_y \forall_z (x - y) \cdot z = x \cdot z - y \cdot z$
(h)	pm0	$\forall_x x \cdot 0 = 0$
(i)	po	$\forall_x x + -x = 0$
(j)	cpa	$\forall_x \forall_y x + y = y + x$
(k)	ps	$\forall_x \forall_y x - y = x + -y$
(l)	pm1	$\forall_x x \cdot 1 = x$
(m)	dpma	$\forall_x \forall_y \forall_z (x + y) \cdot z = x \cdot z + y \cdot z$

Turn to PAGE 47.

[RR]

[Page 47]

JIM is not a very helpful machine if all he can do is show the generalizations. But JIM can do more than that. Remember, he is a Jiffy Instance Maker.

Suppose you push the button marked:

pm1

Here is what JIM's window looks like:

$$\forall_x x \cdot 1 = x$$

Now, if you push the button marked:

INS

JIM erases the quantifying phrase ' \forall_x ' and copies of the related letter 'x'.

So, JIM's window now looks like this.

$$\blacksquare \cdot 1 = \blacksquare$$

If you then write '5' on JIM's blackboard, he will write a copy of this numeral in each place from which he erased an 'x' and JIM's window would then look like this:

$$5 \cdot 1 = 5$$

In JIM's window now is the instance of ' $\forall_x x \cdot 1 = x$ ' which corresponds with '5'.

On your work sheet show what JIM's window would look like if

you pushed

pa0

?

and then pushed

INS

?

and then wrote '-19' on JIM's blackboard.

?

[RR]

[Page 48]

pa0	$\forall_x \quad x + 0 = x$	(a)
INS	$\blacksquare + 0 = \blacksquare$	(b)
'-19'	$-19 + 0 = -19$	(c)

['-19 + 0 = -19' is the instance of ' $\forall_x \quad x + 0 = x$ ' which corresponds with '-19'.]

* * *

Show what JIM's window looks like after you pushed

po	?
INS	?

and then wrote ' $\sqrt{2}$ ' on the blackboard.

?

[RR]

[Page 49]

po $\forall_x x + -x = 0$

INS $\blacksquare + -\blacksquare = 0$

$\sqrt{2}$ $\sqrt{2} + -\sqrt{2} = 0$

* * *

JIM is not limited to copying simple numerals like '5', '-19', and ' $\sqrt{2}$ '. He can also make the instance of ' $\forall_x x \cdot 1 = x$ ' which corresponds with ' $[(7 + 2) \times -9]$ '. Here is what the window would look like after you pushed

pm1, INS, and wrote ' $[(7 + 2) \times -9]$ ' on the blackboard.

$[(7 + 2)x - 9] \cdot 1 = [(7 + 2)x - 9]$

On your work sheet show what JIM's window would look like after you

pushed

and

wrote:

pa0, INS

$(\frac{1}{2} \times \sqrt{7})$

[RR]

[Page 50]

$$\left(\frac{1}{2} \times \sqrt{7}\right) + 0 = \left(\frac{1}{2} \times \sqrt{7}\right)$$

* * *

On your work sheet show what JIM's window would look like if you

pushed

and

wrote

(a) (pml) , (INS)

$$[(8 \div 3) \times 4]$$

(b) (po) , (INS)

$$-5$$

(c) (pa0) , (INS)

$$(8 - 13)$$

(d) (pm0) , (INS)

$$-19$$

[Page 51a]

[RR]

pushedwrotewindow

- | | | | | |
|-----|-------|-----|-------------------------|---|
| (a) | pm1 , | INS | $[(8 \div 3) \times 4]$ | $[(8 \div 3) \times 4] \cdot 1 = [(8 \div 3) \times 4]$ |
| (b) | po , | INS | -5 | $-5 + --5 = 0$ |
| (c) | pa0 , | INS | $(8 - 13)$ | $(8 - 13) + 0 = (8 - 13)$ |
| (d) | pm0 , | INS | -19 | $-19 \cdot 0 = 0$ |

* * *

At times, we may want instances of generalizations other than those which are principles. Here is how we can get JIM to write such instances for us. When you push the button marked 'INS' JIM gets ready to write an instance of the generalization which is in the window. Now, if we push INS when there is not a universal generalization in the window, JIM doesn't do anything. We must first give JIM a universal generalization. One way to do this is by calling out from JIM's memory one of our principles. Here is another way to give JIM a universal generalization to work on.

If you write:

$$\forall_x x \cdot (x + 1) = x \cdot x + x$$

on the blackboard JIM begins to "read" this and as soon as he sees 'V' he knows that a universal generalization is coming. He copies the generalization from the blackboard into the window, and is then ready to act when you push the instance button.

[RR]

[Page 51]



Read the facing page first.

Here is how you could get the instance of ' $\forall_x x \cdot (x + 1) = x \cdot x + x$ ' which corresponds with ' $\frac{2}{3}$ '. Write ' $\forall_x x \cdot (x + 1) = x \cdot x + x$ ' on JIM's blackboard. He will copy it into the window and then erase the blackboard.

Step 1. [You do it.]

$$\forall_x x \cdot (x + 1) = x \cdot x + x$$

Blackboard

Window

Step 2. [JIM does it.]

$$\forall_x x \cdot (x + 1) = x \cdot x + x$$

Step 3. [You push INS, JIM erases.]

$$\blacksquare \cdot (\blacksquare + 1) = \blacksquare \cdot \blacksquare + \blacksquare$$

Step 4. [Write ' $\frac{2}{3}$ ' on the blackboard.]

$$\frac{2}{3}$$

$$\blacksquare \cdot (\blacksquare + 1) = \blacksquare \cdot \blacksquare + \blacksquare$$

Step 5. [JIM writes the instance.]

$$\frac{2}{3} \cdot \left(\frac{2}{3} + 1\right) = \frac{2}{3} \cdot \frac{2}{3} + \frac{2}{3}$$

' $\frac{2}{3} \cdot (\frac{2}{3} + 1) = \frac{2}{3} \cdot \frac{2}{3} + \frac{2}{3}$ ' is the $[\frac{2}{3}]$ -instance of ' $\forall_x x \cdot (x + 1) = x \cdot x + x$ '.

Write, on your work sheet, the $[3]$ -instance of ' $\forall_x x \cdot (x + 1) = x \cdot x + x$ '.

[RR]

[Page 52]

The [3]-instance of ' $\forall_x x \cdot (x + 1) = x \cdot x + x$ '; that is, the instance of ' $\forall_x x \cdot (x + 1) = x \cdot x + x$ ' which corresponds with '3'.

$$3(3 + 1) = 3 \cdot 3 + 3$$

* * *

Write on your work sheet the instance of ' $\forall_x x \cdot (x + 1) = x \cdot x + x$ ' which corresponds with:

(a) 0

(b) $(\frac{2}{3} + 8)$

(c) -17.3

[RR]

[Page 53]

Instances of ' $\forall_x x \cdot (x + 1) = x \cdot x + x$ '

(a) $0 \cdot (0 + 1) = 0 \cdot 0 + 0$

(b) $(\frac{2}{3} + 8) \cdot [(\frac{2}{3} + 8) + 1] = (\frac{2}{3} + 8) \cdot (\frac{2}{3} + 8) + (\frac{2}{3} + 8)$

(c) $-17.3(-17.3 + 1) = -17.3 \cdot -17.3 + -17.3$

* * *

Write ' $\forall_w 1 + 2 \cdot w = 3 \cdot w$ ' on JIM's blackboard. [Of course, you can't really do this, but use your imagination.] Now, on your work sheet, write the instances of ' $\forall_w 1 + 2 \cdot w = 3 \cdot w$ ' which JIM would write if you write on the blackboard:

(a) 3

(b) 1

(c) 3.5

(d) -7

[RR]

[Page 54]

Instances of:

$$\forall_w 1 + 2 \cdot w = 3 \cdot w$$

$$(a) 1 + 2 \cdot 3 = 3 \cdot 3$$

$$(b) 1 + 2 \cdot 1 = 3 \cdot 1$$

$$(c) 1 + 2 \cdot 3.5 = 3 \cdot 3.5$$

$$(d) 1 + 2 \cdot -7 = 3 \cdot -7$$

* * *

Here are some of the rules which JIM follows (and which we will also follow) when writing generalizations or instances.

(1) Only numerals and the letters:

$$w, x, y, z, w_1, x_1, y_1, z_1,$$

are used to write universal generalizations.

(2) Numerals, names for numbers, such as:

$$-7, \sqrt{2}, (5 \div 3), [(8 \times 7) + -3]$$

are used to write instances.

Consider the instances (a), (b), (c), and (d) above.

(1) Write 'T' on your work sheet for each instance which you think is true.

(2) Write 'C' on your work sheet for each instance which you think is a consequence of ' $\forall_w 1 + 2 \cdot w = 3 \cdot w$ '.

[RR]

[Page 55]

Each true instance is printed in green. 'C' is written before each instance which is a consequence of ' $\forall_w 1 + 2 \cdot w = 3 \cdot w$ '.

$$C \quad (a) \quad 1 + 2 \cdot 3 = 3 \cdot 3$$

$$C \quad (b) \quad 1 + 2 \cdot 1 = 3 \cdot 1$$

$$C \quad (c) \quad 1 + 2(3.5) = 3(3.5)$$

$$C \quad (d) \quad 1 + 2(-7) = 3(-7)$$

Each instance (even a false instance) is a consequence of the generalization.

* * *

You probably do not want to accept the universal generalization:

$$\forall_w 1 + 2 \cdot w = 3 \cdot w$$

because you have seen that at least one of its consequences, the instance:

$$1 + 2 \cdot 3 = 3 \cdot 3$$

is false, since $1 + 2 \cdot 3 = 7$, $3 \cdot 3 = 9$, and $7 \neq 9$. However, JIM does not know how to compute and does not concern himself with truth. JIM's main function is to produce instances of generalizations. We can use JIM to produce instances and then we will decide whether to accept the generalization.

Suppose that JIM has given you these sentences as instances of a generalization:

$$(1) \quad -3 \cdot \frac{1}{2} = -3 \div 2$$

$$(2) \quad 8 \cdot \frac{1}{2} = 8 \div 2$$

$$(3) \quad 17 \cdot \frac{1}{2} = 17 \div 2$$

and that:

$$(4) \quad 10 \cdot \frac{1}{2} = 15 \div 2$$

is not an instance of that generalization. Write, on your work sheet, a generalization which JIM may have used to write sentences (1), (2), and (3), but not (4).

[RR]

[Page 56]

$$\forall_x x \cdot \frac{1}{2} = x \div 2$$

* * *

JDM reads only sentences written in our formal language. Translate to a universal generalization.

Pick a number. Multiply 2 by that number, add 1 to the product, and then multiply the sum by 3. The result is the same as the result of multiplying 6 by the number and then adding 3.

[RR]

[Page 57]

$$\forall x (2 \cdot x + 1) \cdot 3 = 6 \cdot x + 3$$

* * *

Translate each of these to a universal generalization.

- (a) Pick a number. Add 1 to the number and then multiply the sum by itself. The result is the product of the chosen number by itself, plus the product of 2 by the chosen number, plus 1.
- (b) Pick any number. Multiply 10 by the number and add 5. Multiply the sum by itself. The result is the same as the result of these operations: add 1 to the number; multiply the sum by the chosen number; multiply by 100; add 25.

[RR]

[Page 58]

$$(a) \forall_y (y + 1) \cdot (y + 1) = y \cdot y + 2 \cdot y + 1$$

$$(b) \forall_z (10 \cdot z + 5) \cdot (10 \cdot z + 5) = (z + 1) \cdot z \cdot 100 + 25$$

* * *

Write the [Z]-instance of:

$$\forall_z (10 \cdot z + 5) \cdot (10 \cdot z + 5) = (z + 1) \cdot z \cdot 100 + 25$$

[If you have any trouble, use JIM!]

[RR]

[Page 59]

$$(10 \cdot 2 + 5)(10 \cdot 2 + 5) = (2 + 1) \cdot 2 \cdot 100 + 25$$

* * *

Remember JIM doesn't care what a sentence "says" so long as it has the right form . But we care. JIM can help us by producing instances. Write the instance of:

$$\forall_z (10z + 5)(10z + 5) = (z + 1) \cdot z \cdot 100 + 25$$

which corresponds with:

(a) 1

(b) 4

(c) 7

[RR]

[Page 60]

$$(a) (10 \cdot 1 + 5) \cdot (10 \cdot 1 + 5) = (1 + 1) \cdot 1 \cdot 100 + 25$$

$$(b) (10 \cdot 4 + 5) \cdot (10 \cdot 4 + 5) = (4 + 1) \cdot 4 \cdot 100 + 25$$

$$(c) (10 \cdot 7 + 5) \cdot (10 \cdot 7 + 5) = (7 + 1) \cdot 7 \cdot 100 + 25$$

* * *

Recall how JIM works. If the generalization $\forall_x x \cdot 1 = x$ is in the window when you push the instance button **INS**, he erases the quantifying phrase \forall_x and each occurrence of the related letter 'x', and you would then see $\square \cdot 1 = \square$ in the window.

If you then write '-86' on JIM's blackboard he will write a copy of '-86' in each place from which he erased an 'x'. Thus, JIM will produce:

$$-86 \cdot 1 = -86$$

as the $[-86]$ -instance of $\forall_x x \cdot 1 = x$.

Suppose the commutative principle for multiplication:

$$\forall_x \forall_y x \cdot y = y \cdot x$$

is in the window when you push **INS**. JIM erases \forall_x and each occurrence of 'x' and this is what you see:

$$\forall_y \square \cdot y = y \cdot \square$$

If you write '5' on the blackboard, JIM will complete $\forall_y \square \cdot y = y \cdot \square$ to look like:

$$\forall_y 5 \cdot y = y \cdot 5$$

Thus, the instance of

$$\forall_x \forall_y x \cdot y = y \cdot x$$

which corresponds with 5 is

$$\forall_y 5 \cdot y = y \cdot 5$$

Write the instance of

$$\forall_x \forall_y x \cdot y = y \cdot x$$

which corresponds with '8'.

[RR]

[Page 61]

The [8]-instance of $\forall_x \forall_y x \cdot y = y \cdot x$

$$\forall_y 8 \cdot y = y \cdot 8$$

* * *

If you push **INS** when:

$$\forall_y 8 \cdot y = y \cdot 8$$

is in the window and then write '13' on the blackboard, JIM will produce:

$$8 \cdot 13 = 13 \cdot 8$$

Since the sentence ' $8 \cdot 13 = 13 \cdot 8$ ' can be obtained from ' $\forall_x \forall_y x \cdot y = y \cdot x$ ' by using JIM, the Jiffy Instance Maker, we will also call:

$$8 \cdot 13 = 13 \cdot 8$$

an instance of:

$$\forall_x \forall_y x \cdot y = y \cdot x$$

In fact, we will call:

$$8 \cdot 13 = 13 \cdot 8$$

the [8, 13]-instance of:

$$\forall_x \forall_y x \cdot y = y \cdot x$$

[Note. We are extending the use of the word 'instance' to include sentences which we can get from JIM by more than one use of the **INS** button.]

Write, on your work sheet, the [7, -2]-instance of ' $\forall_x \forall_y x \cdot y = y \cdot x$ ' and the [-2, 7]-instance of ' $\forall_x \forall_y x \cdot y = y \cdot x$ '.

[RR]

[Page 62]

(a) The $[7, -2]$ -instance of ' $\forall_x \forall_y x \cdot y = y \cdot x$ '

$$7 \cdot -2 = -2 \cdot 7$$

(b) The $[-2, 7]$ -instance of ' $\forall_x \forall_y x \cdot y = y \cdot x$ '

$$-2 \cdot 7 = 7 \cdot -2$$

* * *

Here is the $[5, (7 + 3)]$ -instance of ' $\forall_x \forall_y x + y = y + x$ ':

$$5 + (7 + 3) = (7 + 3) + 5$$

Write, on your work sheet, the $[3, (4 \cdot 8)]$ -instance of:

$$\forall_x \forall_y x + y = y + x$$

[RR]

[Page 63]

The $[3, (4 \cdot 8)]$ -instance of ' $\forall_x \forall_y x + y = y + x$ ':

$$3 + (4 \cdot 8) = (4 \cdot 8) + 3$$

* * *

Here is an instance of one of our principles:

$$(1) \quad \underline{17} + \underline{-17} = 0$$

Write, on your work sheet, the principle of which sentence (1) is an instance.

[The chart in the back of the book may help you.]

[RR]

[Page 64]

' $17 + -17 = 0$ ' is an instance of:

$$\forall_x x + -x = 0$$

* * *

For each of the sentences given below, name the principle of which it is an instance, and state the principle in formal language.

Sample. $3 + 5 = 5 + 3$

Solution. $\text{cpa } \forall_x \forall_y x + y = y + x$

(a) $(7 + -2) \cdot 8 = 7 \cdot 8 + -2 \cdot 8$

(b) $1 \cdot (-5 + 2) = (-5 + 2) \cdot 1$

(c) $17 + -17 = 0$

(d) $(3 + 4.2) + 5 = 3 + (4.2 + 5)$

(e) $\frac{43}{5} + 0 = \frac{43}{5}$

(f) $17.3 + 32 = 32 + 17.3$

[RR]

[Page 65]

(a) $(7 + -2) \cdot 8 = 7 \cdot 8 + -2 \cdot 8$

dpm $\forall_x \forall_y \forall_z (x+y)z = xz + yz$

(c) $17 + -17 = 0$

po $\forall_x x + -x = 0$

(e) $\frac{43}{5} + 0 = \frac{43}{5}$

po $\forall_x x + 0 = x$

(b) $1 \cdot (-5 + 2) = (-5 + 2) \cdot 1$

cpm $\forall_x \forall_y xy = yx$

(d) $(3 + 4.2) + 5 = 3 + (4.2 + 5)$

apa $\forall_x \forall_y \forall_z (x+y)+z = x+(y+z)$

(f) $17.3 + 32 = 32 + 17.3$

cpa $\forall_x \forall_y x+y = y+x$

* * *

For each of these sentences, tell the basic principle of which it is an instance.

(a) $7 - 9 = 7 + -9$

(b) $(47 \cdot 3) \cdot 1 = 47 \cdot 3$

(c) $(46 - 9) \cdot 5 = 46 \cdot 5 - 9 \cdot 5$

(d) $((87 \cdot 13) \cdot -7) \cdot 0 = 0$

(e) $((16 + 5) \cdot 4) \cdot 8 = (16 + 5) \cdot (4 \cdot 8)$

(f) $(-23 + 23) \cdot 0 = -23 \cdot 0 + 23 \cdot 0$

[RR]

[Page 66]

(a) $7 - 9 = 7 + -9$

ps $\forall_x \forall_y x - y = x + -y$

(b) $(47 \cdot 3) \cdot 1 = (47 \cdot 3)$

pm1 $\forall_x x \cdot 1 = x$

(c) $(49 - 9) \cdot 5 = 46 \cdot 5 - 9 \cdot 5$

dpms $\forall_x \forall_y \forall_z (x - y)z = x \cdot z - y \cdot z$

(d) $((87 \cdot 13) \cdot -7) \cdot 0 = 0$

pm0 $\forall_x x \cdot 0 = 0$

(e) $((16 + 5) \cdot 4) \cdot 8 = (16 + 5) \cdot (4 \cdot 8)$ (f) $(-23 + 23) \cdot 0 = -23 \cdot 0 + 23 \cdot 0$

apm $\forall_x \forall_y \forall_z (x \cdot y)z = x \cdot (y \cdot z)$

dpma $\forall_x \forall_y \forall_z (x + y) \cdot z = x \cdot z + y \cdot z$

* * *

When we say that we accept a universal generalization, such as

' $\forall_x \forall_y x - y = x + -y$ ', we mean that we shall thereby accept as true any statements like:

$7 - 9 = 7 + -9$

$-7 - 13 = -7 + -13$

which are instances of the generalization.

We write:

(*) The sentence ' $7 - 9 = 7 + -9$ ' is a consequence of ' $\forall_x \forall_y x - y = x + -y$ '

A convenient translation of (*) is:

$$\frac{\forall_x \forall_y x - y = x + -y}{7 - 9 = 7 + -9}$$

We sometimes say that we infer the sentence ' $7 - 9 = 7 + -9$ ' from the sentence ' $\forall_x \forall_y x - y = x + -y$ ' and we call the figure displayed above an inference.

Translate the sentence:

' $(47 \cdot 3) \cdot 1 = (47 \cdot 3)$ ' is a consequence of ' $\forall_x x \cdot 1 = x$ '

by using a '_____ ' as we did in translating (*).

[RR]

[Page 67]

A translation of the sentence:

' $(47 \cdot 3) \cdot 1 = (47 \cdot 3)$ ' is a consequence of ' $\forall_x x \cdot 1 = x$ '

is:

$$\frac{\forall_x x \cdot 1 = x}{(47 \cdot 3) \cdot 1 = 47 \cdot 3}$$

* * *

Some of these sentences are instances of our principles. For each sentence which is an instance of a principle show that the sentence is a consequence of the basic principle by using a '_____ '.

(a) $5 + -5 = 0$

(b) $(3 + 5) + 7 = 3 + (7 + 5)$

(c) $(9 \cdot 17) + 36 = 36 + (9 \cdot 17)$

(d) $3(2 + 2) = 2(3 + 3)$

[RR]

[Page 68]

$$(a) \frac{\forall x \quad x + -x = 0}{5 + -5 = 0}$$

$$(c) \frac{\forall x \forall y \quad x + y = y + x}{(9 \cdot 17) + 36 = 36 + (9 \cdot 17)}$$

* * *

The rule of reasoning which justifies our acceptance of inferences like:

$$(a) \frac{\forall x \quad x + -x = 0}{5 + -5 = 0}$$

is called universal instantiation.

Universal Instantiation
Each instance of a universal generalization
is a consequence of it.

The sentence:

$$\forall x \quad x + -x = 0$$

is called the premiss of the inference

and the sentence:

$$5 + -5 = 0$$

is called the conclusion of the inference.

Each of the following exercises refers to an example of inferring a conclusion from a premiss by means of universal instantiation. Your job is to complete each example.

$$(a) \frac{\forall x \forall y \quad ?}{5 + 7 = 7 + 5}$$

$$(b) \frac{\forall x \forall y \forall z \quad (x + y) + z = x + (y + z)}{(3 + ?) + 7 = ? + (5 + ?)}$$

$$(c) \frac{\forall x \forall y \quad \text{if } x + y = 0 \text{ then } x = -y}{\text{if } ? \quad \text{then } -3 = -?}$$

$$(d) \frac{\forall x \forall y \forall z \quad ?}{\text{if } 5 + 3 = 8 \text{ then } 5 = 8 - 3}$$

[RR]

[Page 69]

$$(a) \frac{\forall_x \forall_y \ x + y = y + x}{5 + 7 = 7 + 5}$$

$$(b) \frac{\forall_x \forall_y \forall_z \ (x + y) + z = x + (y + z)}{(3 + 5) + 7 = 3 + (5 + 7)}$$

$$(c) \frac{\forall_x \forall_y \ \text{if } x + y = 0 \text{ then } x = -y}{\text{if } -3 + +3 = 0 \text{ then } -3 = -+3}$$

$$(d) \frac{\forall_x \forall_y \forall_z \ \text{if } x + y = z \text{ then } x = z - y}{\text{if } 5 + 3 = 8 \text{ then } 5 = 8 - 3}$$

* * *

We decided that the sentence:

$$(*) \ (3 + 5) + 7 = 3 + (7 + 5)$$

is not an instance of one of our principles. However, we can show that (*) is a consequence of instances of our principles.

The sentence (1) ' $(3 + 5) + 7 = 3 + (5 + 7)$ ' is an instance of one of our principles, namely, ' $\forall_x \forall_y \forall_z \ (x + y) + z = x + (y + z)$ ' [apa]. Now, the sentence:

$$(1) \ (3 + 5) + 7 = 3 + (5 + 7)$$

looks like the sentence (*) except that (*) has ' $7 + 5$ ' where (1) has ' $5 + 7$ '.

Recall, now, that we have the $[5, 7]$ -instance of ' $\forall_x \forall_y \ x + y = y + x$ ' [cpa]; that is, the sentence ' $5 + 7 = 7 + 5$ '. By the sentence ' $5 + 7 = 7 + 5$ ' we mean that the numerals ' $5 + 7$ ' and ' $7 + 5$ ' are names for the same number. Hence, in (1), we may replace a numeral ' $5 + 7$ ' by a numeral ' $7 + 5$ '. Thus, we can infer:

$$(*) \ (3 + 5) + 7 = 3 + (7 + 5)$$

from the sentences:

$$(1) \ (3 + 5) + 7 = 3 + (5 + 7)$$

$$(2) \ 5 + 7 = 7 + 5$$

Here is how we show that (*) is a consequence of (1) and (2).

$$\frac{5 + 7 = 7 + 5 \quad (3 + 5) + 7 = 3 + (5 + 7)}{(3 + 5) + 7 = 3 + (7 + 5)}$$

Complete this inference:

$$\frac{20 + 2 = 22 \quad (20 + 2) \times 5 = 20 \times 5 + 2 \times 5}{?}$$

[RR]

[Page 70]

$$\frac{20 + 2 = 22 \quad (20 + 2) \times 5 = 20 \times 5 + 2 \times 5}{22 \times 5 = 20 \times 5 + 2 \times 5}$$

* * *

Here is another inference, one premiss of which is an equation.

$$\frac{5 = 3 + 2 \quad 5 > 3}{3 + 2 > 3}$$

Note that the other premiss in this inference is not an equation.

Complete each of these inferences on your work sheet.

(a) $\frac{3 + 4 = 7 \quad 3 \cdot 5 + 4 \cdot 5 = (3 + 4) \cdot 5}{?}$

(b) $\frac{8 + (-3) = (6 + 2) + (-3) \quad 6 > 8 + (-3)}{?}$

(c) $\frac{\begin{array}{l} \text{My Alma Mater} \\ \text{is} \\ \text{Zabbranchburg University.} \end{array} \quad \begin{array}{l} \text{My Alma Mater} \\ \text{is in Lewistown.} \end{array}}{?}$

[RR]

[Page 71]

$$(a) \quad \frac{3 + 4 = 7 \qquad 3 \cdot 5 + 4 \cdot 5 = (3 + 4) \cdot 5}{3 \cdot 5 + 4 \cdot 5 = 7 \cdot 5}$$

$$(b) \quad \frac{8 + (-3) = (6 + 2) + (-3) \qquad 6 > 8 + (-3)}{6 > (6 + 2) + (-3)}$$

$$(c) \quad \frac{\begin{array}{c} \text{My Alma Mater} \\ \text{is} \\ \text{Zabbranchburg University.} \end{array} \qquad \begin{array}{c} \text{My Alma Mater} \\ \text{is in Lewistown.} \end{array}}{\text{Zabbranchburg University is in Lewiston.}}$$

* * *

We indicate that it is correct to make the inference:

$$\frac{3 + 4 = 7 \qquad 3 \cdot 5 + 4 \cdot 5 = (3 + 4)5}{3 \cdot 5 + 4 \cdot 5 = 7 \cdot 5}$$

by saying that the inference is valid. Inference (b) is also valid. Is inference (c) valid? Suppose you find out that Zabbranchburg University is not in Lewiston. Would you conclude that incorrect reasoning was used in (c)? Most people would say something like:

The reasoning is correct, but one of the premisses must be false.

Decide which of these inferences are valid. Indicate your decisions by writing 'VALID' or 'INVALID' (in-VAL-id) on your work sheet.

$$(a) \quad \frac{\begin{array}{c} \text{Los Angeles} \\ \text{is} \\ \text{the capital of California} \end{array} \qquad \begin{array}{c} \text{Reno, Nevada} \\ \text{is east of} \\ \text{Los Angeles} \end{array}}{\text{Reno, Nevada is east of the capital of California}}$$

$$(b) \quad \frac{1 = 2 \qquad 2 + 7 = 13}{1 + 7 = 13}$$

$$(c) \quad \frac{5 + 3 = 8 \qquad (5 + 3) + 2 = 10}{8 + 3 = 11}$$

[RR]

[Page 72]

- (a)
$$\frac{\begin{array}{c} \text{Los Angeles} \\ \text{is} \\ \text{the capital of California} \end{array} \quad \text{VALID} \quad \begin{array}{c} \text{Reno, Nevada} \\ \text{is east of} \\ \text{Los Angeles} \end{array}}{\text{Reno, Nevada is east of the capital of California}}$$
- (b)
$$\frac{1 = 2 \quad \text{VALID} \quad 2 + 7 = 13}{1 + 7 = 13}$$
- (c)
$$\frac{5 + 3 = 8 \quad \text{INVALID} \quad (5 + 3) + 2 = 10}{8 + 3 = 11}$$

* * *

In inference (a), both premisses are false [check this with an atlas], the conclusion is true, and the inference is valid. In inference (b), both premisses are false, the conclusion is false, and the inference is valid. In inference (c), both premisses are true, the conclusion is true, and the reasoning is not valid (invalid). A person who uses '=' the way we do and who knew nothing about arithmetic would make the inference given in (b), but would not make the inference given in (c). At this point in our discussion, we are interested in valid reasoning and not in the truth or falsity of the premisses or conclusion.

Decide which of the statements (a) - (e) are true and which are false. If you decide a statement is false, do not write it on your work sheet. If you decide a statement is true, write it on your work sheet.

- (a) Any inference with true premisses is valid.
- (b) Any inference with false premisses is invalid.
- (c) Any inference with a true conclusion is valid.
- (d) Any inference with a false conclusion is invalid.
- (e) The validity of an inference is independent of the truth of its premisses or the truth of its conclusion.

[RR]

[Page 73]

The statement:

The validity of an inference is independent of the truth
of its premisses or the truth of its conclusion.

is true.

The other statements (a) - (d) are false.

* * *

Here is a valid inference:

$$\frac{6 + 2 = 8 \qquad 7(6 + 2) > 5(6 + 2)}{7(8) > 5(6 + 2)}$$

Here is another with the same premisses:

$$\frac{6 + 2 = 8 \qquad 7(6 + 2) > 5(6 + 2)}{7(6 + 2) > 5(8)}$$

Complete the inference patterns given below. Although the premisses are the same, you should write different conclusions for the three examples.

$$(a) \quad \frac{3 + 1 = 4 \qquad (3 + 1) + 7 > (3 + 1) + 6}{?}$$

$$(b) \quad \frac{3 + 1 = 4 \qquad (3 + 1) + 7 > (3 + 1) + 6}{?}$$

$$(c) \quad \frac{3 + 1 = 4 \qquad (3 + 1) + 7 > (3 + 1) + 6}{?}$$

[RR]

[Page 74]

$$(a) \quad \frac{3 + 1 = 4 \qquad (3 + 1) + 7 > (3 + 1) + 6}{4 + 7 > (3 + 1) + 6}$$

$$(b) \quad \frac{3 + 1 = 4 \qquad (3 + 1) + 7 > (3 + 1) + 6}{(3 + 1) + 7 > 4 + 6}$$

$$(c) \quad \frac{3 + 1 = 4 \qquad (3 + 1) + 7 > (3 + 1) + 6}{4 + 7 > 4 + 6}$$

* * *

Which of these inferences are valid? Write, on your work sheet, either 'valid' or 'invalid' according to your decision.

$$(a) \quad \frac{9 \cdot 8 = 72 \qquad 9 \cdot 8 + 1 \cdot 8 = (9 + 1)8}{72 + 1 \cdot 8 = (9 + 1)8}$$

$$(b) \quad \frac{836 \times 425 = 355,200 \qquad 836 \times 425 = 425 \times 836}{355,200 = 425 \times 836}$$

$$(c) \quad \frac{7 \times 3 \times (-862) = -18,102 \qquad (7 \times 3) \times (-862) = 3 \times (7 \times -862)}{(7 \times 3) \times (-862) = 21 \times (-862)}$$

$$(d) \quad \frac{4 \cdot 2 = 8 \qquad 4 \cdot 2 + 9 = (4 \cdot 2 + 7) + 2}{4 \cdot 2 + 9 = (8 + 7) + 2}$$

$$(a) \quad \frac{9 \cdot 8 = 72 \quad \text{VALID} \quad 9 \cdot 8 + 1 \cdot 8 = (9 + 1)8}{72 + 1 \cdot 8 = (9 + 1)8}$$

$$(b) \quad \frac{836 \times 425 = 355,200 \quad \text{VALID} \quad 836 \times 425 = 425 \times 836}{355,200 = 425 \times 836}$$

$$(c) \quad \frac{7 \times 3 \times (-862) = -18,102 \quad \text{VALID} \quad (7 \times 3) \times (-862) = 3 \times (7 \times -862)}{(7 \times 3) \times (-862) = 3 \times (7 \times -862)}$$

$$(d) \quad \frac{4 \cdot 2 = 8 \quad \text{VALID} \quad 4 \cdot 2 + 9 = (4 \cdot 2 + 7) + 2}{4 \cdot 2 + 9 = (8 + 7) + 2}$$

* * *

Here is the rule of reasoning which justifies inferences like (a), (b) and (d).

The Replacement Rule for Equations

Given an equation and another sentence, if one side of the equation is replaced by the other side somewhere in the sentence, the new sentence thus obtained is a consequence of the given equation and sentence.

[RR]

[Page 75]



Read the facing page first.

Let's see how this rule of reasoning applies to the inference:

$$\frac{3 = 2 + 1 \quad 3 > 5}{2 + 1 > 5}$$

Given an equation and another sentence,

$$3 = 2 + 1 \quad 3 > 5$$

if one side of the equation is replaced by the other side

$$\xrightarrow{\quad} 3 = 2 + 1 \xleftarrow{\quad}$$

somewhere in the sentence,

$$\frac{3 = 2 + 1 \quad 3 > 5}{2 + 1 > 5}$$

the new sentence thus obtained is a consequence of the given equation and sentence.

Each of the exercises below refers to an example of inferring a conclusion from two premisses by means of the replacement rule for equations. Your job is to find a sentence which completes the pattern.

$$(i) \quad \frac{5 + -5 = 0 \quad 3(5 + -5) + 7 = 7}{?}$$

$$(ii) \quad \frac{\frac{1}{2} + -\frac{1}{2} = 0 \quad ?}{3 \cdot 0 + 7 = 7}$$

$$(iii) \quad \frac{? \quad 3(\sqrt{10} + -\sqrt{10}) + 7 = 7}{3 \cdot 0 + 7 = 7}$$

$$(*) \quad \frac{a + -a = 0 \quad 3(a + -a) + 7 = 7}{?}$$

$$(i) \quad \frac{5 + -5 = 0}{3 \cdot 0 + 7 = 7} \quad \frac{3(5 + -5) + 7 = 7}{3 \cdot 0 + 7 = 7}$$

$$(ii) \quad \frac{\frac{1}{2} + -\frac{1}{2} = 0}{3 \cdot 0 + 7 = 7} \quad \frac{3(\frac{1}{2} + -\frac{1}{2}) + 7 = 7}{3 \cdot 0 + 7 = 7}$$

$$(iii) \quad \frac{\sqrt{10} + -\sqrt{10} = 0}{3 \cdot 0 + 7 = 7} \quad \frac{3(\sqrt{10} + -\sqrt{10}) + 7 = 7}{3 \cdot 0 + 7 = 7}$$

$$(*) \quad \frac{a + -a = 0}{3 \cdot 0 + 7 = 7} \quad \frac{3(a + -a) + 7 = 7}{3 \cdot 0 + 7 = 7}$$

[RR]

[Page 76]

[Answers are on the facing page.]

You may have noticed that:

$$(*) \quad \frac{a + -a = 0}{3 \cdot 0 + 7 = 7} \quad \frac{3(a + -a) + 7 = 7}{}$$

is a pattern which can be used to get each of the valid inferences (i), (ii), and (iii). For example, if you substitute a '5' for each occurrence of 'a' in the pattern (*), you get:

$$(i) \quad \frac{5 + -5 = 0}{3 \cdot 0 + 7 = 7} \quad \frac{3(5 + -5) + 7 = 7}{}$$

What should you substitute for 'a' in the pattern (*) to get the valid inference (ii)?

More generally, the pattern:

$$(**) \quad \frac{a + -a = 0}{b \cdot 0 + c = c} \quad \frac{b(a + -a) + c = c}{}$$

can be used to generate valid inferences by substituting numerals for the letters. [You must substitute a copy of the same numeral for each occurrence of a particular letter, say 'a', throughout the pattern. Contrast this with replacement.] Since substitution of numerals in the pattern (**) always results in valid inferences, we will call the pattern itself a valid inference. Whenever we have a pattern from which only valid inferences can be obtained, we will call the pattern a valid inference. Hence, we say that both (*) and (**) are valid inferences.

Complete these sentences on your work sheet.

- (a) To get the inference (i) from the pattern (**), substitute ' ____ ? ____ ' for 'a', ' ____ ? ____ ' for 'b', and ' ____ ? ____ ' for 'c'
- (b) To get the inference (ii) from the pattern (**), substitute ' ____ ? ____ ' for 'a', ' ____ ? ____ ' for 'b', and ' ____ ? ____ ' for 'c'

[RR]

[Page 77]

(a) To get the inference:

$$\frac{5 + -5 = 0 \quad 3(5 + -5) + 7 = 7}{3 \cdot 0 + 7 = 7}$$

from:

$$(**) \quad \frac{a + -a = 0 \quad b(a + -a) + c = c}{b \cdot 0 + c = c}$$

substitute '5' for 'a', '3' for 'b', and '7' for 'c'.

(b) To get the inference:

$$\frac{\frac{1}{2} + -\frac{1}{2} = 0 \quad 3(\frac{1}{2} + -\frac{1}{2}) + 7 = 7}{3 \cdot 0 + 7 = 7}$$

use (**) and substitute ' $\frac{1}{2}$ ' for 'a', '3' for 'b', and '7' for 'c'.

* * *

Use the replacement rule for equations and complete these to valid inferences.

$$(a) \quad \frac{4 + 1 = 5 \quad 8(4 + 1) > ?}{8 \cdot 5 > 7(4 + 1)}$$

$$(b) \quad \frac{b = 5 - a \quad 2b + 3a = 7}{?}$$

$$(c) \quad \frac{b = d \quad \text{if } a > b \text{ and } ?}{\text{if } a > d \text{ and } d > c \text{ then } a > c}$$

$$(d) \quad \frac{7 = ? \quad 7 + 3 > 9}{(6 + 1) + 3 > 9}$$

[RR]

[Page 78]

(a) $8(4 + 1) > 7(4 + 1)$

(b) $2(5 - a) + 3a = 7$

(c) if $a > b$ and $b > c$ then $a > c$

(d) $7 = 6 + 1$

* * *

We collected several valid inferences to show the use of the replacement rule for equations. However, someone erased part of each inference before we could copy them into this book. Here is what we have left:

(a)
$$\frac{\text{[erased]} \quad 5b + 3b = 8b}{5 \cdot 7 + 3 \cdot 7 = 8 \cdot 7}$$

(b)
$$\frac{a + 0 = a \quad a(a + 0) = aa + a0}{\text{[erased]}}$$

(c)
$$\frac{a = b \quad b = b}{\text{[erased]}}$$

(d)
$$\frac{a = b \quad b = a}{\text{[erased]}}$$

(e)
$$\frac{\text{[erased]} \quad \text{if } a = b \text{ then } a + c = b + c}{\text{if } a = b \text{ then } a + d = b + d}$$

(f)
$$\frac{\text{Sacramento is the capital of California} \quad \text{James lives in Sacramento}}{\text{[erased]}}$$

You can use what is left to write valid inferences which are justified by the replacement rule for equations. Here is what (a) may have looked like before erasure:

$$\frac{b = 7 \quad 5b + 3b = 8b}{5 \cdot 7 + 3 \cdot 7 = 8 \cdot 7}$$

or it might have looked like this:

$$\frac{7 = b \quad 5b + 3b = 8b}{5 \cdot 7 + 3 \cdot 7 = 8 \cdot 7}$$

Show, on your work sheet, what (a) - (f) might have looked like before erasure.

[Page 79a]

[RR]

$$(a) \frac{b = 7 \quad 5b + 3b = 8b}{5 \cdot 7 + 3 \cdot 7 = 8 \cdot 7}$$

$$(b) \frac{a + 0 = a \quad a(a + 0) = aa + a0}{aa = aa + a0}$$

$$(c) \frac{a = b \quad b = b}{b = a}$$

$$(d) \frac{a = b \quad b = a}{a = a}$$

$$(e) \frac{c = d \quad \text{if } a = b \text{ then } a + c = b + c}{\text{if } a = b \text{ then } a + d = b + d}$$

$$(f) \frac{\begin{array}{l} \text{Sacramento is} \\ \text{the capital of California} \end{array} \quad \begin{array}{l} \text{James lives} \\ \text{in Sacramento} \end{array}}{\text{James lives in the capital of California}}$$

For examples (a) - (e) there is at least one sentence other than the one given above which might have been used. For (a), '7 = b' might be the sentence which was erased.

for (b) '(a + 0)(a + 0) = aa + a0'

or 'a(a + 0) = a(a + 0) + a0'

or '(a + 0)a = a(a + 0) + a0'

or any other sentence obtained by replacing any (at least one) of the five 'a's in the second premiss by 'a + 0'

for (c) 'a = a' or 'a = b'

(d) 'b = b'

(e) 'd = c'

You are correct if you used some of these sentences instead of the ones we used.

* * *

[RR]

[Page 79]

[Answers are on the facing page.]

We decided earlier that the sentence:

$$(3 + 5) + 7 = 3 + (7 + 5)$$

is not an instance of one of our principles. However, instances are not the only sentences which are consequences of (follow from) our principles. Read again the discussion on page 69 which shows how the sentence ' $(3 + 5) + 7 = 3 + (7 + 5)$ ' can be derived from instances of our principles. We want to justify the sentence ' $(3 + 5) + 7 = 3 + (7 + 5)$ ' without doing any computing. We note that the sentence looks something like an instance of the associative principle for addition. However, if we write the instance of the apa which starts ' $(3 + 5) + 7 =$ ', we get:

$$(1) \quad (3 + 5) + 7 = 3 + (5 + 7)$$

and we want:

$$(3) \quad (3 + 5) + 7 = 3 + (7 + 5)$$

But we also know:

$$(2) \quad 5 + 7 = 7 + 5$$

Hence, from (1), (2), and the replacement rule for equations, we get (3). Here is the inference:

$$\frac{5 + 7 = 7 + 5 \qquad (3 + 5) + 7 = 3 + (5 + 7)}{(3 + 5) + 7 = 3 + (7 + 5)}$$

This inference shows that (3) is a consequence of (1) and (2).

Write, on your work sheet, an inference which shows that the sentence

$$2(7 + 1) = 2 \cdot 7 + 2$$

is a consequence of the sentences:

$$2 \cdot 1 = 2$$

$$2(7 + 1) = 2 \cdot 7 + 2 \cdot 1$$

[RR]

[Page 80]

$$\begin{array}{ccc}
 (1) & & (2) \\
 \frac{2 \cdot 1 = 2}{2(7+1) = 2 \cdot 7 + 2} & \frac{2(7+1) = 2 \cdot 7 + 2 \cdot 1}{2(7+1) = 2 \cdot 7 + 2} & \\
 (3) & &
 \end{array}$$

* * *

The inference given above shows that ' $2(7+1) = 2 \cdot 7 + 2$ ' is a consequence of the sentences ' $2 \cdot 1 = 2$ ' and ' $2(7+1) = 2 \cdot 7 + 2 \cdot 1$ '. Since ' $2 \cdot 1 = 2$ ' is a consequence of ' $\forall_x x \cdot 1 = x$ ' and the sentence ' $2(7+1) = 2 \cdot 7 + 2 \cdot 1$ ' is a consequence of ' $\forall_x \forall_y \forall_z x(y+z) = xy + xz$ ', we say that ' $2(7+1) = 2 \cdot 7 + 2$ ' is a consequence of ' $\forall_x x \cdot 1 = x$ ' and ' $\forall_x \forall_y \forall_z x(y+z) = xy + xz$ '. Here is the complete derivation:

$$(*) \quad \frac{\frac{\forall_x x \cdot 1 = x}{2 \cdot 1 = 2} \quad \frac{\forall_x \forall_y \forall_z x(y+z) = xy + xz}{2(7+1) = 2 \cdot 7 + 2 \cdot 1}}{2(7+1) = 2 \cdot 7 + 2}$$

The derivation is made up of three inferences:

$$\begin{array}{ll}
 (1) \quad \frac{\forall_x x \cdot 1 = x}{2 \cdot 1 = 2} & (2) \quad \frac{\forall_x \forall_y \forall_z x(y+z) = xy + xz}{2(7+1) = 2 \cdot 7 + 2 \cdot 1} \\
 (3) \quad \frac{2 \cdot 1 = 2 \quad 2(7+1) = 2 \cdot 7 + 2 \cdot 1}{2(7+1) = 2 \cdot 7 + 2} &
 \end{array}$$

Each of the inferences (1), (2), and (3) is valid. Hence, we say that the derivation (*) is valid.

The rule of reasoning which justifies inference (1) is universal instantiation. Complete these sentences on your work sheet.

- (a) Inference (2) is justified by _____.
 (b) Inference (3) is justified by _____.

[RR]

[Page 81]

- (a) universal instantiation
- (b) the replacement rule for equations

* * *

Most people use 'For each x ' in such a way that if they accept, for example, the sentence ' $\forall x \ 1 \cdot x = x$ ' they are agreeing to accept each of its instances, such as:

$$(1) \ 1 \cdot 5 = 5 \qquad (2) \ 1(-7) = -7 \qquad (3) \ 1\left(\frac{2}{3}\right) = \frac{2}{3}$$

and, if they accept individually each instance of a generalization (they should live so long), then they would accept the generalization. Now we have agreed to accept certain universal generalizations which we call basic principles. How about the generalization ' $\forall x \ 1 \cdot x = x$ '? Should we add this to our list of our accepted generalizations?

You probably accept the sentences (1) - (3) on the basis of "computing facts" which you have learned. But let's pretend that the only things we know about operations with numbers are the things which are consequences of our principles. Can we derive the sentence ' $1 \cdot 5 = 5$ ' from our principles? The answer is 'Yes', and here is an informal derivation:

$$(a) \ 1 \cdot 5 = 5 \cdot 1 \text{ [cpm]} \qquad (b) \ 5 \cdot 1 = 5 \text{ [pml]} \qquad \text{Hence, (c) } 1 \cdot 5 = 5$$

So, if you accept the cpm, the pml, and certain rules of reasoning, you must accept the sentence ' $1 \cdot 5 = 5$ '. In order to see which rules of reasoning we used we will write a formal derivation.

Here is how we show that (a) is a consequence of the cpm.

$$\frac{\forall x \forall y \ xy = yx}{1 \cdot 5 = 5 \cdot 1}$$

Write, on your work sheet, an inference which shows that step (b) is a consequence of the pml.

[RR]

[Page 82]

$$\frac{\forall_x x \cdot 1 = x}{5 \cdot 1 = 5}$$

* * *

Step (c) follows from step (a) and step (b) by the replacement rule for equations.

$$\frac{\begin{array}{c} (b) \\ 5 \cdot 1 = 5 \end{array} \quad \begin{array}{c} (a) \\ 1 \cdot 5 = 5 \cdot 1 \end{array}}{1 \cdot 5 = 5} \quad (c)$$

Now, if we put the three inferences together

$$\frac{\frac{\forall_x x \cdot 1 = x}{5 \cdot 1 = 5} \quad \frac{\forall_x \forall_y xy = yx}{1 \cdot 5 = 5 \cdot 1}}{1 \cdot 5 = 5}$$

we have a derivation in which the conclusion is ' $1 \cdot 5 = 5$ ' and the premisses are ' $\forall_x x \cdot 1 = x$ ' and ' $\forall_x \forall_y xy = yx$ '. This derivation shows that the sentence ' $1 \cdot 5 = 5$ ' is a consequence of ' $\forall_x x \cdot 1 = x$ ' and ' $\forall_x \forall_y xy = yx$ '. Since we have agreed to accept our principles and consequences of our principles, we should accept the sentence ' $1 \cdot 5 = 5$ ' without relying on "computing facts".

What rule of reasoning justifies the inference $\frac{\forall_x x \cdot 1 = x}{5 \cdot 1 = 5}$?

[RR]

[Page 83]

The rule of reasoning which justifies the inference:

$$\frac{\forall x \ x \cdot 1 = x}{5 \cdot 1 = 5}$$

is universal instantiation.

* * *

We can show the rule of reasoning which justifies each inference in our derivation by making notations next to the horizontal lines. For example, we can write:

$$\frac{\forall x \ x \cdot 1 = x}{5 \cdot 1 = 5} \text{ UI}$$

Next to each horizontal line in the derivation on your work sheet show the rule of reasoning which justifies the inference.

[RR]

[Page 84]

$$\begin{array}{c}
 \frac{\forall x \quad x \cdot 1 = x}{5 \cdot 1 = 5} \text{ UI} \qquad \frac{\forall x \forall y \quad xy = yx}{1 \cdot 5 = 5 \cdot 1} \text{ UI} \\
 \hline
 1 \cdot 5 = 5 \qquad \text{RRE}
 \end{array}$$

* * *

Since each inference in our derivation is valid (justified by one of our rules of reasoning), we say that the derivation is valid.

Write a derivation which shows that the sentence ' $1 \cdot -7 = -7$ ' is a consequence of ' $\forall x \quad x \cdot 1 = x$ ' and ' $\forall x \forall y \quad xy = yx$ '. Put a box \square around each ' -7 ' so that you won't be tempted to use any computing facts. Also, at the end of each line in the derivation, indicate the rule of reasoning which justifies the inference.

[Page 85a]

[RR]

$$\begin{array}{c}
 \frac{\forall_x x \cdot 1 = x}{\boxed{-7} \cdot 1 = \boxed{-7}} \text{UI} \quad \frac{\forall_x \forall_y xy = yx}{1 \cdot \boxed{-7} = \boxed{-7} \cdot 1} \text{UI} \\
 \hline
 1 \cdot \boxed{-7} = \boxed{-7} \quad \text{RRE}
 \end{array}$$

* * *

Do you see an easy way to test the $\left[\frac{2}{3}\right]$ -instance of ' $\forall_x 1 \cdot x = x$ '? Simply erase each '-7' in the derivation you wrote on your work sheet and write a ' $\frac{2}{3}$ ' in each "box". After you erase each '-7' you have:

$$\begin{array}{c}
 \frac{\forall_x x \cdot 1 = x}{\boxed{} \cdot 1 = \boxed{}} \text{UI} \quad \frac{\forall_x \forall_y xy = yx}{1 \cdot \boxed{} = \boxed{} \cdot 1} \text{UI} \\
 (*) \quad \hline
 1 \cdot \boxed{} = \boxed{} \quad \text{RRE}
 \end{array}$$

(*) is a test-pattern which can be used to test and verify any instance of ' $\forall_x 1 \cdot x = x$ '. Note that we do not claim that the test-pattern can be used in verifying every instance. Life is too short! However, if you accept the pml ' $\forall_x x \cdot 1 = x$ ' and the cpm ' $\forall_x \forall_y xy = yx$ ', it is unreasonable for you to doubt any instance. If someone should claim, for example, that -19 "doesn't work" all you have to do is write a '-19' in each box and you have a derivation which shows that the $[-19]$ -instance of ' $\forall_x 1 \cdot x = x$ ' is a consequence of the pml and the cpm.

[RR]

[Page 85]

[Read the facing page first.]

Because the generalization ' $\forall_x 1 \cdot x = x$ ' has infinitely many instances, we cannot derive every instance from the generalizations ' $\forall_x x \cdot 1 = x$ ' and ' $\forall_x \forall_y xy = yx$ '. But the pattern

$$\begin{array}{c}
 \frac{\forall_x x \cdot 1 = x}{\boxed{} \cdot 1 = \boxed{}} \text{UI} \quad \frac{\forall_x \forall_y xy = yx}{1 \cdot \boxed{} = \boxed{} \cdot 1} \text{UI} \\
 \hline
 1 \cdot \boxed{} = \boxed{} \quad \text{RRE}
 \end{array}$$

shows us how we can derive any instance of ' $\forall_x 1 \cdot x = x$ ' which we want. It thus seems reasonable to accept the test pattern itself as evidence that ' $\forall_x 1 \cdot x = x$ ' is a consequence of the pm1 and the cpm.

We indicate this acceptance by writing

$$\begin{array}{c}
 \frac{\forall_x x \cdot 1 = x}{\boxed{} \cdot 1 = \boxed{}} \text{UI} \quad \frac{\forall_x \forall_y xy = yx}{1 \cdot \boxed{} = \boxed{} \cdot 1} \text{UI} \\
 \hline
 1 \cdot \boxed{} = \boxed{} \quad \text{RRE} \\
 \hline
 \forall_x 1 \cdot x = x
 \end{array}$$

Note the "double line" drawn below the test pattern.

Turn to PAGE 86.

[RR]

[Page 86]

Here is a test-pattern for instances of the generalization ' $\forall_y 0 + y = y$ '.

$$(**) \quad \frac{\frac{\forall_x \forall_y x + y = y + x}{a + 0 = 0 + a} \text{ UI} \quad \frac{\forall_x x + 0 = x}{a + 0 = a} \text{ UI}}{0 + a = a} \text{ RRE}$$

Note that we have used 'a' [instead of ' \square '] as a pattern variable. From now on, we will use the letters 'a', 'b', 'c', and 'd' as pattern variables. Since we have a test-pattern which can be used to test and verify any instance of ' $\forall_y 0 + y = y$ ' we will say:

' $\forall_y 0 + y = y$ ' is a consequence of ' $\forall_x \forall_y x + y = y + x$ ' and ' $\forall_x x + 0 = x$ '

and show this by:

$$\frac{\frac{\forall_x \forall_y x + y = y + x}{a + 0 = 0 + a} \text{ UI} \quad \frac{\forall_x x + 0 = x}{a + 0 = a} \text{ UI}}{0 + a = a} \text{ RRE} \\ \hline \forall_y 0 + y = y$$

The rule of reasoning which justifies the last step is called the test-pattern principle (TPP).

Test Pattern Principle (TPP)

A universal generalization is a consequence of given sentences if there is a pattern for showing that any instance of the generalization is a consequence of the given sentences.

Thus, the universal generalization ' $\forall_y 0 + y = y$ ' is a consequence of the sentences:

$$(1) \quad \forall_x \forall_y x + y = y + x \quad \text{and:} \quad (2) \quad \forall_x x + 0 = x$$

since there is a pattern (**) for showing that any instance of the generalization is a consequence of sentence (1) and (2).

Turn to PAGE 87.

[RR]

[Page 87]

We had written a derivation of the generalization ' $\forall_x 0 \cdot x = 0$ ' but our "eraser-friend" destroyed parts of it. Here is what we have left:

$$\begin{array}{c}
 \frac{\forall_x \forall_y \quad x \cdot y = y \cdot x}{a \cdot 0 = 0 \cdot a} \text{ UI} \qquad \frac{\forall_x \quad x \cdot 0 = 0}{a \cdot 0 = 0} \text{ UI} \\
 \hline
 0 \cdot a = 0 \qquad \qquad \qquad \text{RRE} \\
 \hline
 \qquad \qquad \qquad ? \qquad \qquad \qquad \text{TPP}
 \end{array}$$

Write, on your work sheet, a universal generalization which completes the derivation.

[RR]

[Page 88]

$$\begin{array}{c}
 \frac{\forall x \forall y \ x \cdot y = y \cdot x}{a \cdot 0 = 0 \cdot a} \text{ UI} \\
 \frac{\frac{\frac{\forall x \ x \cdot 0 = 0}{a \cdot 0 = 0} \text{ UI}}{0 \cdot a = 0} \text{ RRE}}{\forall x \ 0 \cdot x = 0} \text{ TPP}
 \end{array}$$

* * *

Show that the derivation given below is valid by giving on your work sheet the rule of reasoning which justifies each inference.

$$\begin{array}{c}
 \frac{\forall x \forall y \ x \cdot y = y \cdot x}{a \cdot 3 = 3 \cdot a} \text{ (1)} \\
 \frac{\frac{\forall x \forall y \forall z \ (x \cdot y) \cdot z = x \cdot (y \cdot z)}{(a \cdot 3) \cdot b = a \cdot (3 \cdot b)} \text{ (2)}}{(3 \cdot a) \cdot b = a \cdot (3 \cdot b)} \text{ (3)} \\
 \frac{\frac{\frac{\frac{\forall x \forall y \ (x \cdot y) \cdot z = x \cdot (y \cdot z)}{(a \cdot 3) \cdot b = a \cdot (3 \cdot b)} \text{ (2)}}{(3 \cdot a) \cdot b = a \cdot (3 \cdot b)} \text{ (3)}}{\forall x \forall y \ (3 \cdot x) \cdot y = x \cdot (3 \cdot y)} \text{ (4)}
 \end{array}$$

[Page 89a]

[RR]

(1) UI

(2) UI

(3) RRE

(4) TPP

* * *

We hope that you have had no trouble in seeing which rules of inference justify the steps in our derivations. However, you may be wondering if you could write a complete derivation by yourself.

It is a very unusual person indeed who can sit down and immediately write out a complete derivation for a generalization. Most of us sketch out an abbreviated plan for a derivation which we feel we could expand to a complete derivation if asked to.

Here is such a plan for a derivation of the generalization:

$$\forall_x 2 \cdot (x + 1) = 2 \cdot x + 2$$

If we could write a test pattern in which the last line is ' $2 \cdot (a + 1) = 2 \cdot a + 2$ ' we could use the test pattern principle to get ' $\forall_x 2 \cdot (x + 1) = 2 \cdot x + 2$ '. Our job then is to start with ' $2 \cdot (a + 1)$ ' and transform it to ' $2 \cdot a + 2$ '. Now, if we had ' $(a + 1) \cdot 2$ ' instead of ' $2 \cdot (a + 1)$ ', we could apply the dpma. But ' $2 \cdot (a + 1) = (a + 1) \cdot 2$ ' is an instance of cpm. So we write:

$$(1) \quad 2 \cdot (a + 1) = (a + 1) \cdot 2 \quad \text{cpm}$$

$$(2) \quad (a + 1) \cdot 2 = a \cdot 2 + 1 \cdot 2 \quad \text{dpma}$$

Now, we note that we have ' $1 \cdot 2$ ' and we want ' 2 ', and we also see that the [2] - instance of the pml would be helpful here. So we write:

$$(3) \quad 2 \cdot 1 = 2$$

But we don't have ' $2 \cdot 1$ ', we have ' $1 \cdot 2$ '. Ah, the cpm will help again.

$$(4) \quad 1 \cdot 2 = 2 \cdot 1$$

[RR]

[Page 89]

Here is a formal column derivation which shows the reasoning involved in the steps.

(1)	$2 \cdot (a + 1) = (a + 1) \cdot 2$	$\forall_x \forall_y x \cdot y = y \cdot x$	UI
		$\forall_x \forall_y \forall_z (x + y) \cdot z = x \cdot z + y \cdot z$?
(2)	$(a + 1) \cdot 2 = a \cdot 2 + 1 \cdot 2$		
(3)	$1 \cdot 2 = 2 \cdot 1$	$\forall_x \forall_y x \cdot y = y \cdot x$?
(4)	$2 \cdot 1 = 2$	$\forall_x x \cdot 1 = x$?
(5)	$1 \cdot 2 = 2$		(4), (3); RRE
(6)	$(a + 1) \cdot 2 = a \cdot 2 + 2$		(5), (2) ?
(7)	$a \cdot 2 = 2 \cdot a$	$\forall_x \forall_y x \cdot y = y \cdot x$?
(8)	$(a + 1) \cdot 2 = 2 \cdot a + 2$		(7), (6) ?
(9)	$2 \cdot (a + 1) = 2 \cdot a + 2$		(8), (1) ?
(10)	$\forall_x 2 \cdot (x + 1) = 2 \cdot x + 2$		(1) - (9) ?

On your work sheet, give the rule of reasoning which justifies each step. Some of these have already been given.

[RR]

[Page 90]

(2) UI

(3) UI

(4) UI

(6) RRE

(7) UI

(8) RRE

(9) RRE

(10) TPP

* * *

Each of the lines in the column derivation on page 89 corresponds with an inference. For example, line (1) corresponds with the inference:

$$\frac{\forall_x \forall_y \ x \cdot y = y \cdot x}{2 \cdot (a+1) = (a+1) \cdot 2} \text{ UI}$$

Line (5) corresponds to the inference:

$$\frac{2 \cdot 1 = 2 \quad 1 \cdot 2 = 2 \cdot 1}{1 \cdot 2 = 2} \text{ RRE}$$

On your work sheet, write the inference which corresponds with

(a) line (4)

(b) line (8)

[RR]

[Page 91]

$$(a) \quad \frac{\forall x \quad x \cdot 1 = x}{2 \cdot 1 = 2} \text{ UI}$$

$$(b) \quad \frac{a \cdot 2 = 2 \cdot a \quad (a+1) \cdot 2 = a \cdot 2 + 2}{(a+1) \cdot 2 = 2 \cdot a + 2} \text{ RRE}$$

* * *

Here is a tree-diagram of the derivation on page 89. On your work sheet, show a rule of reasoning for each inference.

$$\begin{array}{c}
 \frac{\text{cpm}}{(1)} \quad ? \quad \frac{\text{cpm}}{(7)} \quad ? \quad \frac{\text{pm1}}{(4)} \quad ? \quad \frac{\text{cpm}}{(3)} \quad ? \quad \frac{\text{dpma}}{(2)} \quad ? \\
 \hline
 \frac{(5)}{(6)} \quad ? \\
 \hline
 \frac{(8)}{(9)} \quad ? \\
 \hline
 \forall x \quad 2 \cdot (x+1) = 2 \cdot x + 2 \quad ?
 \end{array}$$

[RR]

[Page 92]

		$\frac{\text{pm1}}{(4)} \text{ UI}$	$\frac{\text{cpm}}{(3)} \text{ UI}$	$\frac{\text{dpma}}{(2)}$	UI
	$\frac{\text{cpm}}{(7)} \text{ UI}$	(5)	(6)		RRE
$\frac{\text{cpm}}{(1)} \text{ UI}$		(8)			RRE
		(9)			RRE
$\forall_x 2 \cdot (x+1) = 2 \cdot x + 2$					TPP

* * *

The steps of the column derivation which are given at the top of each "branch" in the "tree" above are the premisses of the derivation and the sentence at the "base" of the tree is the conclusion of the derivation. Hence, one of the premisses of the derivation is:

$$\forall_x \forall_y xy = yx \quad [\text{cpm}]$$

Complete on your work sheet.

(a) The premisses of the derivation given on Page 89 are:

(i) $\forall_x \forall_y xy = yx \quad [\text{cpm}]$

(ii) ?

(iii) ?

(b) The conclusion of the derivation given on Page 89 is:

?

[RR]

[Page 93]

(a) The premisses of the derivation are:

$$(1) \quad \forall_x \forall_y \quad x \cdot y = y \cdot x$$

$$(2) \quad \forall_x \forall_y \forall_z \quad (x + y) \cdot z = x \cdot z + y \cdot z$$

$$(3) \quad \forall_x \quad x \cdot 1 = x$$

(b) The conclusion of the derivation is:

$$\forall_x \quad 2 \cdot (x + 1) = 2 \cdot x + 2$$

* * *

In the example on page 89 each sentence is an instance of a universal generalization or follows from earlier steps by application of one of our rules of inference [universal instantiation (UI), the replacement rule for equations (RRE), or the test pattern principle (TPP)]. Such a sequence of sentences is called a valid derivation of the conclusion.

When, as in this case, the premisses:

$$\forall_x \forall_y \quad x \cdot y = y \cdot x$$

$$\forall_x \forall_y \forall_z \quad (x + y) \cdot z = x \cdot z + y \cdot z$$

$$\forall_x \quad x \cdot 1 = x$$

are principles, the conclusion is called a theorem.

Turn to Page 94.

[RR]

[Page 94]

Is the sentence ' $\forall_x \forall_y x \cdot (y \cdot 1) = x \cdot y$ ' a theorem? We can show that it is a theorem if we give a derivation in which the sentence

$$\forall_x \forall_y x \cdot (y \cdot 1) = x \cdot y$$

is the conclusion and the premisses are basic principles. Here is a column derivation of ' $\forall_x \forall_y x \cdot (y \cdot 1) = x \cdot y$ '.

	$\forall_x \forall_y \forall_z (x \cdot y) \cdot z = x \cdot (y \cdot z)$	
(1)	$(a \cdot b) \cdot 1 = a \cdot (b \cdot 1)$	UI
(2)	$(a \cdot b) \cdot 1 = a \cdot b$	$\forall_x x \cdot 1 = x$ UI
•		
(3)	$a \cdot (b \cdot 1) = a \cdot b$	(1), (2); RRE
(4)	$\forall_x \forall_y x \cdot (y \cdot 1) = x \cdot y$	(1) - (3); TPP

Write, on your work sheet, the corresponding tree-diagram.

[RR]

[Page 95]

$$\begin{array}{l}
 \frac{\frac{\frac{\forall x \forall y \forall z (x \cdot y) \cdot z = x \cdot (y \cdot z)}{\quad} \quad \frac{\forall x x \cdot 1 = x}{\quad}}{\frac{(a \cdot b) \cdot 1 = a \cdot (b \cdot 1)}{\quad} \quad (a \cdot b) \cdot 1 = a \cdot b} \text{UI} \quad \text{UI} \\
 \frac{a \cdot (b \cdot 1) = a \cdot b}{\quad} \text{RRE} \\
 \frac{\quad}{\forall x \forall y x \cdot (y \cdot 1) = x \cdot y} \text{TPP}
 \end{array}$$

* * *

Complete on your work sheet.

(a) The premisses of the derivation above are:

?

(b) The conclusion of the derivation above is:

?

[RR]

[Page 96]

(a) The premisses of the derivation are:

$$\forall_x \forall_y \forall_z (x \cdot y) \cdot z = x \cdot (y \cdot z)$$

$$\forall_x x \cdot 1 = x$$

(b) The conclusion of the derivation is:

$$\forall_x \forall_y x \cdot (y \cdot 1) = x \cdot y$$

* * *

Since the premisses of the derivation are principles, the derivation shows that the conclusion is a theorem.

Complete on your work sheet.

(a) A statement which is a logical consequence of the basic principles is _____?

(b) In a derivation which shows that a sentence is a theorem, the basic principles are the _____? of the derivation and the given sentence is the _____? of the derivation.

[RR]

[Page 97]

- (a) A statement which is a logical consequence of the basic principles is a theorem.
- (b) In a derivation which shows that a sentence is a theorem, the basic principles are the premisses of the derivation and the given sentence is the conclusion of the derivation.

* * *

On your work sheet, write a derivation of the sentence:

$$\forall x \forall y \ x \cdot (y \cdot 0) = 0$$

Use only principles as premisses. You may write either a column or a tree-diagram. [Hint: Try apm and pm0.]

[RR]

[Page 98]

	$\forall_x \forall_y \forall_z (x \cdot y) \cdot z = x \cdot (y \cdot z)$	
(1)	$(a \cdot b) \cdot 0 = a \cdot (b \cdot 0)$	UI
(2)	$(a \cdot b) \cdot 0 = 0$	$\forall_x x \cdot 0 = 0$ UI
(3)	$a \cdot (b \cdot 0) = 0$	(1), (2); RRE
(4)	$\forall_x \forall_y x \cdot (y \cdot 0) = 0$	(1) - (3); TPP

OR

$\forall_x \forall_y \forall_z (x \cdot y) \cdot z = x \cdot (y \cdot z)$		$\forall_x x \cdot 0 = 0$	
$(a \cdot b) \cdot 0 = a \cdot (b \cdot 0)$	UI	$(a \cdot b) \cdot 0 = 0$	UI
$a \cdot (b \cdot 0) = 0$			RRE
$\forall_x \forall_y x \cdot (y \cdot 0) = 0$			TPP

* * *

Show that the sentence:

$$\forall_x \forall_y x + (y + 0) = x + y$$

is a theorem. Write a complete derivation like one of those above. Remember, use only principles as premisses.

[RR]

[Page 99]

- (1) $(a + b) + 0 = a + b$ $\forall_x x + 0 = x$ UI
-
- (2) $(a + b) + 0 = a + (b + 0)$ $\forall_x \forall_y \forall_z (x + y) + z = x + (y + z)$ UI
-
- (3) $a + (b + 0) = a + b$ (2), (1); RRE
-
- (4) $\forall_x \forall_y x + (y + 0) = x + y$ (1) - (3); TPP

OR

$$\begin{array}{rcl}
 \forall_x \forall_y \forall_z (x + y) + z = x + (y + z) & & \forall_x x + 0 = x \\
 \hline
 (a + b) + 0 = a + (b + 0) & \text{UI} & \\
 \hline
 a + (b + 0) = a + b & & \forall_x x + 0 = x \\
 \hline
 \forall_x \forall_y x + (y + 0) = x + y & & (a + b) + 0 = a + b \\
 \hline
 & & \text{UI} \\
 & & \text{RRE} \\
 & & \text{TPP}
 \end{array}$$

* * *

In these 99 pages we have discussed universal generalizations and some rules of reasoning which are used in derivations. The rules of reasoning are:

Universal Instantiation	[UI]	Page 68
Replacement Rule for Equations	[RRE]	Page 75a
Test Pattern Principle	[TPP]	Page 86

We have seen how a conclusion may be obtained from premisses by applying one or more of these rules. We have agreed that any inference which is an example of the use of one of these rules is valid and that a derivation is valid if it is made up of valid inferences. It would be helpful if you would start a list of rules of inference by copying UI, RRE, and TPP on a separate sheet of paper.

Turn to PAGE 100.

[RR]

[Page 100]

QUIZ

A. Here are several valid inferences. For each inference, give the rule of reasoning which justifies that inference.

$$(a) \frac{\forall x \forall y \ x \cdot y = y \cdot x}{9 \cdot \frac{3}{5} = \frac{3}{5} \cdot 9} ?$$

$$(b) \frac{5+2=7 \quad 5+2>5}{7>5} ?$$

$$(c) \frac{\forall x \ x+7=13}{9+7=13} ?$$

$$(d) \frac{\frac{\forall x \forall y \ x+y=y+x}{3 \cdot a+7=7+3 \cdot a} ?}{\forall x \ 3 \cdot x+7=7+3 \cdot x} ?$$

B. Which sentences are the premisses in this derivation?

- | | | | |
|-----|---------------------|---------------------------------|---------------|
| (1) | $a+0=0+a$ | $\forall x \forall y \ x+y=y+x$ | UI |
| (2) | $a+0=a$ | $\forall x \ x+0=x$ | UI |
| (3) | $0+a=a$ | | (1), (2); RRE |
| (4) | $\forall x \ 0+x=x$ | | (1)-(3); TPP |

C. Complete to true sentences.

1. An inference which is justified by one of our rules of reasoning is called a ? inference.
2. The last line in a derivation is called the ? of the derivation.
3. The sentence ' $3+4=4+3$ ' is an ? of the universal generalization ' $\forall x \forall y \ x+y=y+x$ '.
4. A universal generalization which tells you the same thing as the sentence:

No matter what number you pick, the result
of multiplying that number by 1 is that number

is: ?

The answers to this quiz are on PAGE 101a.

[Page 101a]

[RR]

A. (a) Universal Instantiation

(b) Replacement Rule for Equations

(c) Universal Instantiation

(d) Universal Instantiation
Test Pattern PrincipleB. $\forall_x \forall_y x + y = y + x; \forall_x x + 0 = x$

C. 1. valid 2. conclusion

3. instance 4. $\forall_x x \cdot 1 = x$

* * *

If you had any wrong answers, review the first 99 pages and then go on to Page 101.

If all your answers were correct, go on to Page 101.

[RR]

[Page 101]

We have accepted some universal generalizations about numbers. We have also accepted some rules of reasoning (UI, RRE, and TPP). We will use these generalizations and rules of reasoning to derive other sentences about numbers.

The rule of reasoning called Universal Instantiation (UI) tells you that each instance of a universal generalization is a logical consequence of the generalization. Universal Instantiation (UI) is the rule of reasoning which justifies our saying that the inference:

$$\frac{\forall x \forall y \quad xy = yx}{9 \times \frac{3}{5} = \frac{3}{5} \times 9}$$

is valid.

Here is another valid inference. Which rule of reasoning justifies this inference?

$$\frac{9 \times \frac{3}{5} = \frac{3}{5} \times 9 \quad 5 \times (9 \times \frac{3}{5}) = 5 \times (\frac{3}{5} \times 9)}{5 \times (9 \times \frac{3}{5}) = 5 \times (\frac{3}{5} \times 9)}$$

Write your answer on your work sheet.

[RR]

[Page 102]

The rule of reasoning which justifies the inference:

$$\frac{9 \times \frac{3}{5} = \frac{3}{5} \times 9 \qquad 5 \times (9 \times \frac{3}{5}) = 5 \times (9 \times \frac{3}{5})}{5 \times (9 \times \frac{3}{5}) = 5 \times (\frac{3}{5} \times 9)}$$

is the replacement rule for equations. (RRE)

* * *

We can use the rules of inference to derive sentences from our principles.

Here is an informal derivation of the sentence:

$$(1) \quad 5 \times (9 \times \frac{3}{5}) = (5 \times \frac{3}{5}) \times 9$$

Informal derivation:

$$5 \times (9 \times \frac{3}{5}) = 5 \times (\frac{3}{5} \times 9) \text{ by the cpm.}$$

$$(5 \times \frac{3}{5}) \times 9 = 5 \times (\frac{3}{5} \times 9) \text{ by the apm.}$$

$$\text{Hence,} \quad 5 \times (9 \times \frac{3}{5}) = (5 \times \frac{3}{5}) \times 9$$

Here is a column derivation of (1):

$$(1) \quad 5 \times (9 \times \frac{3}{5}) = 5 \times (9 \times \frac{3}{5})$$

$$(2) \quad 9 \times \frac{3}{5} = \frac{3}{5} \times 9$$

$$(3) \quad 5 \times (9 \times \frac{3}{5}) = 5 \times (\frac{3}{5} \times 9)$$

$$(4) \quad (5 \times \frac{3}{5}) \times 9 = 5 \times (\frac{3}{5} \times 9)$$

$$(5) \quad 5 \times (9 \times \frac{3}{5}) = (5 \times \frac{3}{5}) \times 9$$

$$\forall_x \forall_y \quad xy = yx$$

$$\forall_x \forall_y \forall_z \quad (xy)z = x(yz)$$

UI

(2), (1); RRE

UI

(4), (3); RRE

Turn to PAGE 103.

[RR]

[Page 103]

Line 2 of the derivation is a translation of the inference:

$$\frac{\forall_x \forall_y \quad xy = yx}{9 \times \frac{3}{5} = \frac{3}{5} \times 9} \quad \text{UI}$$

Line 3 of the derivation is a translation of the inference:

$$\frac{9 \times \frac{3}{5} = \frac{3}{5} \times 9 \qquad 5 \times (9 \times \frac{3}{5}) = 5 \times (9 \times \frac{3}{5})}{5 \times (9 \times \frac{3}{5}) = 5 \times (\frac{3}{5} \times 9)} \quad \text{RRE}$$

On your work sheet, write translations for line 4 and line 5 of the derivation.

[RR]

[Page 104]

$$(4) \quad \frac{\forall_x \forall_y \forall_z (xy)z = x(yz)}{(5 \times \frac{3}{5}) \times 9 = 5 \times (\frac{3}{5} \times 9)}$$

$$(5) \quad \frac{(5 \times \frac{3}{5}) \times 9 = 5 \times (\frac{3}{5} \times 9) \qquad 5 \times (9 \times \frac{3}{5}) = 5 \times (\frac{3}{5} \times 9)}{5 \times (9 \times \frac{3}{5}) = (5 \times \frac{3}{5}) \times 9}$$

* * *

Here is a tree-diagram which shows the logical connections among the first three steps of the derivation. On your work sheet, write a tree-diagram for the entire derivation.

$$\frac{\frac{?}{?} \qquad \frac{\frac{\text{cpm}}{(2)} \qquad (1) \text{ RRE}}{(3)} \qquad ?}{?}$$

[RR]

[Page 105]

$$\begin{array}{rcl}
 \frac{\text{apm}}{(4)} & \frac{\frac{\text{cpm}}{(2)}}{(3)} & (1) \\
 \hline
 5 \times (9 \times \frac{3}{5}) & = & (5 \times \frac{3}{5}) \times 9
 \end{array}
 \begin{array}{l}
 \text{RRE} \\
 \text{RRE}
 \end{array}$$

* * *

The conclusion of the derivation above is

$$'5 \times (9 \times \frac{3}{5}) = (5 \times \frac{3}{5}) \times 9'$$

On your work sheet complete this sentence:

The premisses of the derivation on Page 102 are:

- (1) ?
- (2) ?
- (3) ?

[RR]

[Page 106]

The premisses of the derivation on Page 102 are:

$$(1) \quad \forall_x \forall_y \quad x \cdot y = y \cdot x$$

$$(2) \quad \forall_x \forall_y \forall_z \quad (x \cdot y) \cdot z = x \cdot (y \cdot z)$$

$$(3) \quad 5 \times (9 \times \frac{3}{5}) = 5 \times (9 \times \frac{3}{5})$$

* * *

Recall that we accepted the principles [cpm, cpa, apa, etc.] because they summarized facts of arithmetic that we already know. Hence, we took the principles as true sentences about arithmetic and agreed to accept the logical consequences of the principles. The derivation above shows that the sentence ' $5 \times (9 \times \frac{3}{5}) = (5 \times \frac{3}{5}) \times 9$ ' is a consequence of two principles and the sentence:

$$(*) \quad 5 \times (9 \times \frac{3}{5}) = 5 \times (9 \times \frac{3}{5})$$

We should accept (*) without computing. Hence, we should accept the conclusion of the derivation as a true sentence without any reference to computing facts.

More generally, any sentence of the form:

$$\boxed{} = \boxed{}$$

is an acceptable premiss in a derivation.

Law of Identity (LI)

The sentence:

$$\forall_x \quad x = x$$

is an acceptable premiss in a derivation.

Turn to PAGE 107.

[RR]

[Page 107]

We have already written a derivation of the sentence ' $\forall x \forall y x + (y + 0) = x + y$ '.
[See Page 99.] Here is another derivation of the sentence ' $\forall x \forall y x + (y + 0) = x + y$ ':

- | | | | |
|-----|---|-----------------------|----------------|
| (1) | $a + (b + 0) = a + (b + 0)$ | $\forall x x = x$ | UI |
| (2) | $b + 0 = b$ | $\forall x x + 0 = x$ | UI |
| (3) | $a + (b + 0) = a + b$ | | (2), (1); RRE |
| (4) | $\forall x \forall y x + (y + 0) = x + y$ | | (1) - (3); TPP |

Here is the start of a derivation of the sentence ' $\forall x \forall y x \cdot (y \cdot 1) = x \cdot y$ ':

- | | | | |
|-----|---|---------------------------|----|
| (1) | $a \cdot (b \cdot 1) = a \cdot (b \cdot 1)$ | $\forall x x = x$ | UI |
| (2) | $b \cdot 1 = b$ | $\forall x x \cdot 1 = x$ | UI |
| (3) | | | |
| (4) | | | |

On your work sheet complete the derivation.

[RR]

[Page 108]

- (1) $a \cdot (b \cdot 1) = a \cdot (b \cdot 1)$ $\forall_x x = x$ UI
- (2) $b \cdot 1 = b$ $\forall_x x \cdot 1 = x$ UI
- $a \cdot (b \cdot 1) = a \cdot b$ (2), (1); RRE
- $\forall_x \forall_y x \cdot (y \cdot 1) = x \cdot y$ (1) - (3); TPP

* * *

Here is a tree-diagram which corresponds with the derivation given above:

$$\begin{array}{rcl}
 \frac{\forall_x x \cdot 1 = x}{b \cdot 1 = b} & \text{UI} & \frac{\forall_x x = x}{a \cdot (b \cdot 1) = a \cdot (b \cdot 1)} \text{UI} \\
 \hline
 a \cdot (b \cdot 1) = a \cdot b & \text{RRE} & \\
 \hline
 \forall_x \forall_y x \cdot (y \cdot 1) = x \cdot y & \text{TPP} &
 \end{array}$$

On your work sheet, write a tree-diagram of the derivation given on Page 107.

[RR]

[Page 109]

$$\begin{array}{c}
 \frac{\frac{\forall x \quad x + 0 = x}{b + 0 = b} \text{ UI}}{\frac{\frac{\forall x \quad x = x}{a + (b + 0) = a + (b + 0)} \text{ UI}}{a + (b + 0) = a + b} \text{ RRE}} \\
 \hline
 \forall x \forall y \quad x + (y + 0) = x + y \quad \text{TPP}
 \end{array}$$

* * *

Consider these sentences:

$$(1) \quad (5 + 3) - 3 = 5$$

$$(2) \quad (-7 + 2) - 2 = -7$$

$$(3) \quad \left(\frac{1}{2} + -3\right) - (-3) = \frac{1}{2}$$

$$(4) \quad (\sqrt{2} + 5) - 5 = \sqrt{2}$$

Write, on your work sheet, a universal generalization of which each of the sentences (1) - (4) is an instance.

[RR]

[Page 110]

$$\forall_x \forall_y (x + y) - y = x$$

* * *

Given below is an incomplete column derivation of the generalization ' $\forall_x \forall_y (x + y) - y = x$ '. The completed derivation shows that the generalization follows from our principles. More specifically, it follows from four of them.

(1)	$(a + b) - b = (a + b) + -b$	$\forall_x \forall_y x - y = x + -y$	UI
	<hr/>		
	$\forall_x \forall_y \forall_z (x + y) + z = x + (y + z)$		
(2)	?		?
	<hr/>		
(3)	$(a + b) - b = a + (b + -b)$?; RRE
	<hr/>		
(4)	$b + -b = 0$?	?
	<hr/>		
(5)	$(a + b) - b = ?$?
	<hr/>		
(6)	?	$\forall_x x + 0 = x$?
	<hr/>		
(7)	$? = a$		(6), (5); ?
	<hr/>		
(8)	$\forall_x \forall_y (x + y) - y = x$		(1) - (7); ?

On your work sheet, write the complete derivation.

[RR]

[Page 111]

- | | | | |
|-----|---|--------------------------------------|----------------|
| (1) | $(a + b) - b = (a + b) + -b$ | $\forall_x \forall_y x - y = x + -y$ | UI |
| | $\forall_x \forall_y \forall_z (x + y) + z = x + (y + z)$ | | |
| (2) | $(a + b) + -b = a + (b + -b)$ | | UI |
| (3) | $(a + b) - b = a + (b + -b)$ | | (2), (1); RRE |
| (4) | $b + -b = 0$ | $\forall_x x + -x = 0$ | UI |
| (5) | $(a + b) - b = a + 0$ | | (4), (3); RRE |
| (6) | $a + 0 = a$ | $\forall_x x + 0 = x$ | UI |
| (7) | $(a + b) - b = a$ | | (6), (5); RRE |
| (8) | $\forall_x \forall_y (x + y) - y = x$ | | (1) - (7); TPP |

* * *

Write, on your work sheet, a tree-diagram which corresponds with the column derivation given above.

[RR]

[Page 112]

$$\begin{array}{c}
 \frac{\forall_x x+0=x}{a+0=a} \text{UI} \quad \frac{\forall_x x+(-x)=0}{b+(-b)=0} \text{UI} \quad \frac{\forall_x \forall_y \forall_z (x+y)+z=x+(y+z)}{(a+b)+(-b)=a+(b+(-b))} \text{UI} \quad \frac{\forall_x \forall_y x-y=x+(-y)}{(a+b)-b=(a+b)+(-b)} \text{UI} \\
 \frac{(a+b)+(-b)=a+(b+(-b))}{(a+b)-b=a+(b+(-b))} \text{RRE} \\
 \frac{(a+b)-b=a+(b+(-b))}{(a+b)-b=a+0} \text{RRE} \\
 \frac{(a+b)-b=a+0}{(a+b)-b=a} \text{RRE} \\
 \hline
 \forall_x \forall_y (x+y)-y=x \text{TPP}
 \end{array}$$

* * *

In the tree-diagram shown above

- (1) there is exactly one sentence which is not the premiss of any inference.

Any tree diagram (or column of sentences) which satisfies sentence (1) is called a derivation. The sentence which is not a premiss of any inference in the derivation is called the conclusion of the derivation. Any sentence which occurs in the derivation but not as the conclusion of any inference is called a premiss of the derivation.

Complete these sentences on your work sheet.

- (a) The conclusion of the derivation shown above is:

?

- (b) The premisses of the derivation are:

$$\forall_x x+0=x$$

?

?

?

[RR]

[Page 113]

(a) The conclusion of the derivation on Page 112 is:

$$\forall_x \forall_y (x + y) - y = x$$

(b) The premisses of the derivation are:

$$\forall_x x + 0 = x$$

$$\forall_x x + -x = 0$$

$$\forall_x \forall_y \forall_z (x + y) + z = x + (y + z)$$

$$\forall_x \forall_y x - y = x + -y$$

* * *

Recall that each inference which is justified by one of our rules of reasoning is called a valid inference. Each of the inferences in the derivation on Page 112 is justified by one of our rules of reasoning and hence

(2) each inference in the derivation is valid.

Whenever each inference in a derivation is valid, we say that the derivation is valid. A valid derivation shows that the conclusion of the derivation is a consequence of (follows from) the premisses of the derivation.

Complete this sentence on your work sheet.

The valid derivation given on Page 112 shows that the sentence

' _____ ? ' is a consequence of:

?

[RR]

[Page 114]

The valid derivation given on Page 112 shows that the sentence

' $\forall_x \forall_y (x + y) - y = x$ ' is a consequence of:

$$\forall_x x + 0 = x$$

$$\forall_x x + -x = 0$$

$$\forall_x \forall_y \forall_z (x + y) + z = x + (y + z)$$

$$\forall_x \forall_y x - y = x + -y$$

* * *

Write a valid column derivation for the sentence ' $\forall_x 2x + 3x = 5x$ '. You may use the sentence ' $2 + 3 = 5$ ' as one of the premisses. Hint: Derive the pattern sentence ' $5a = 2a + 3a$ '; then use the sentence ' $5a = 5a$ ' and RRE. Start with ' $(2 + 3)a =$ '.

[RR]

[Page 115]

	$\forall_x \forall_y \forall_z (x + y) \cdot z = x \cdot z + y \cdot z$	
(1)	$(2 + 3)a = 2a + 3a$	UI
(2)	$2 + 3 = 5$	Premiss
(3)	$5a = 2a + 3a$	(2), (1); RRE
(4)	$5a = 5a$	$\forall_x x = x$ UI
(5)	$2a + 3a = 5a$	(4), (3); RRE
(6)	$\forall_x 2x + 3x = 5x$	(1) - (5); TPP

* * *

Do we have a valid derivation? We do if our column satisfies conditions (1) and (2).

(1) There is exactly one sentence which is not a premiss of any inference.

(2) Each inference is valid.

On your work sheet, write a tree-diagram which corresponds with the claimed derivation shown above.

[RR]

[Page 116]

$$\begin{array}{c}
 \frac{2+3=5 \quad \frac{\frac{\forall x \forall y \forall z (x+y)z = xz + yz}{(2+3)a = 2a + 3a} \text{ UI}}{5a = 2a + 3a} \text{ RRE} \quad \frac{\forall x x = x}{5a = 5a} \text{ UI} \\
 \hline
 2a + 3a = 5a \quad \text{RRE} \\
 \hline
 \forall x 2x + 3x = 5x \quad \text{TPP}
 \end{array}$$

* * *

Does the tree-diagram shown above satisfy condition (1)?

- (1) There is exactly one sentence which is not the premiss of any inference.

Does the tree-diagram satisfy condition (2)?

- (2) Each inference is valid.

The answer to each question is 'YES'. Hence, we have a valid derivation.

On your work sheet, complete these sentences.

- (a) The conclusion of the derivation on page 116 is: ____?____.
- (b) The premisses of the derivation on page 116 are: ____?____.
- (c) The conclusion of the derivation on page 112 is: ____?____.
- (d) The premisses of the derivation on page 112 are: ____?____.

[RR]

[Page 117]

- (a) The conclusion of the derivation on page 116 is:

$$\forall_x 2x + 3x = 5x$$

- (b) The premisses of the derivation on page 116 are:

$$\forall_x \forall_y \forall_z (x + y)z = xz + yz$$

$$\forall_x x = x$$

$$2 + 3 = 5$$

- (c) The conclusion of the derivation on page 112 is:

$$\forall_x \forall_y (x + y) - y = x$$

- (d) The premisses of the derivation on page 112 are:

$$\forall_x x + 0 = x$$

$$\forall_x x + -x = 0$$

$$\forall_x \forall_y \forall_z (x + y) + z = x + (y + z)$$

$$\forall_x \forall_y x - y = x + -y$$

Complete these sentences on your work sheet.

- (a) The sentence ' $\forall_x 2x + 3x = 5x$ ' is a logical consequence of the sentences: ?

- (b) The sentence ' $\forall_x \forall_y (x + y) - y = x$ ' is a logical consequence of the sentences: ?

$$\nabla_x \nabla_y \nabla_z (x+y)z = xz + yz$$

$$\forall x \quad x = x$$

$$2 + 3 = 5$$

(b) The sentence ' $\forall_x \forall_y (x + y) - y = x$ ' is a logical consequence of:

$$\forall_x \quad x + 0 = x$$

$$\forall_x \quad x + -x = 0$$

$$\forall_x \forall_y \forall_z (x + y) + z = x + (y + z)$$

$$\forall_x \forall_y \quad x - y = x + -y$$

* * *

Some sentences, for example ' $\forall_x \forall_y (x + y) - y = x$ ', are logical consequences of our principles [cpa, dpma, pm0, etc.]. Such sentences are called theorems.

A theorem is a statement which is the conclusion of a valid derivation whose only premisses are generalizations which we have accepted.

Note that although the sentence ' $\forall x \ 2x + 3x = 5x$ ' is the conclusion of a valid derivation, we cannot say that ' $\forall x \ 2x + 3x = 5x$ ' is a theorem since at least one of the premisses in the derivation is not a generalization. [$2 + 3 = 5$]

Complete these sentences on your work sheet.

- (a) If a statement is a logical consequence of our principles, then the statement is _____.
- (b) If a statement is a theorem, then the statement is a logical consequence of _____.

[RR]

[Page 119]

- (a) If a statement is a logical consequence of our principles, then the statement is a theorem.
- (b) If a statement is a theorem, the statement is a logical consequence of generalizations which we have accepted.

* * *

Not all universal generalizations are theorems. Given below are several generalizations. Decide which of these generalizations can be derived from our principles. That is, decide which are theorems. On your work sheet, write the generalizations which are theorems.

(a) $\forall_x \forall_y x(y + 1) = xy + x$

(b) $\forall_x x(2x - 3) = 2xx - 3x$

(c) $\forall_x 1 + 2x = 3x$

(d) $\forall_x 1 \cdot x = x$

(e) $\forall_x \forall_y x(y + y) = y(x + x)$

(f) $\forall_x 2x + 3x = 6xx$

(g) $\forall_x \forall_y x - y = y - x$

[RR]

[Page 120]

The theorems are printed in green.

$$(a) \quad \forall_x \forall_y \quad x(y + 1) = xy + x$$

$$(b) \quad \forall_x \quad x(2x - 3) = 2xx - 3x$$

$$(c) \quad \forall_x \quad 1 + 2x = 3x$$

$$(d) \quad \forall_x \quad 1 \cdot x = x$$

$$(e) \quad \forall_x \forall_y \quad x(y + y) = y(x + x)$$

$$(f) \quad \forall_x \quad 2x + 3x = 6xx$$

$$(g) \quad \forall_x \forall_y \quad x - y = y - x$$

* * *

How can we justify the decisions given above? We have said that the sentence ' $\forall_x \forall_y \quad x(y + 1) = xy + x$ ' is a theorem. Recall our definition of theorem.

A theorem is a statement which is the conclusion of a valid derivation whose only premisses are generalizations which we have accepted.

Hence, in order to show that the sentence ' $\forall_x \forall_y \quad x(y + 1) = xy + x$ ' is a theorem we must give a valid derivation which has only principles or theorems as premisses and the sentence ' $\forall_x \forall_y \quad x(y + 1) = xy + x$ ' as conclusion.

As mentioned earlier, most people write (or think through) an informal derivation of a sentence before attempting to write a complete column or tree-diagram derivation.

You try writing an informal derivation of the sentence ' $\forall_x \forall_y \quad x(y + 1) = xy + x$ '. Do this on "scratch paper" before reading the next page. When you finish your informal derivation, or if you get "stuck", turn to PAGE 121a.

[Page 121a]

[RR]

On this page and on page 121 is a plan which we think can be expanded to a valid derivation of the sentence ' $\forall_x \forall_y x(y + 1) = xy + x$ '. Even if you were able to write what you think is a good plan, you may receive some help by seeing how we "think through" an informal derivation.

One of the more difficult parts in writing a derivation is the first step. You know where you have to start (the principles), and you know where you want to end (the sentence ' $\forall_x \forall_y x(y + 1) = xy + x$ '). But you may feel that "you can't get there from here". The "here" is the set of principles, and the "there" is the sentence ' $\forall_x \forall_y x(y + 1) = xy + x$ '.

Step (1). We note that the conclusion we want:

$$\forall_x \forall_y x(y + 1) = xy + x$$

is a universal generalization. This suggests that we try to derive the pattern sentence:

$$(*) \quad a(b + 1) = ab + a$$

and then we could use TPP to get the universal generalization. But how can we derive the pattern sentence?

Step (2). We notice that the principle operation indicated on the left side of:

$$(*) \quad a(b + 1) = ab + a$$

is multiplication and the principle operation indicated on the right side is addition. This suggests that the distributive principle for multiplication over addition might be helpful. But the sentence (*) is not an instance of the dpma.

Step (3). We cannot use the dpma to transform the expression ' $a(b + 1)$ '. What principle can we use? We can transform the expression ' $a(b + 1)$ ' by using the commutative principle for multiplication. Let's do it.

The appropriate instance of the cpm is:

$$a(b + 1) = (b + 1)a$$

[RR]

[Page 121]

Step (4). Now, we can use the dpma and write:

$$(b + 1)a = ba + 1 \cdot a$$

At this point we are not sure that we are going in the right direction but at least we have done something.

Step (5). From steps (3) and (4) we can get:

$$a(b + 1) = ba + 1 \cdot a$$

by using RRE.

Step (6). Next, we compare the sentence we have derived:

$$a(b + 1) = ba + 1 \cdot a$$

with the sentence we want:

$$a(b + 1) = ab + a$$

We notice that we have 'ba' and we want 'ab'. The cpm and RRE will help again.

Step (7). So, we can get ' $a(b + 1) = ab + 1 \cdot a$ '

Step (8). We have ' $1 \cdot a$ ' and we want ' a '. We can use the cpm and the pml and RRE and get the sentence:

$$a(b + 1) = ab + a$$

Step (9). Then, we will use TPP and get the generalization:

$$\forall_x \forall_y x(y + 1) = xy + x$$

We now have a plan for what we believe is a valid derivation of the sentence ' $\forall_x \forall_y x(y + 1) = xy + x$ '. For many mathematicians the plan is enough and they feel no need to expand the informal derivation into a complete derivation. However, we are amateurs at deriving sentences and we may have made some errors in reasoning which are not apparent in the outline. Let's expand the outline into a complete column so that we can easily check our reasoning.

[RR]

[Page 122]

Here is our column derivation of ' $\forall_x \forall_y x(y + 1) = xy + x$ ':

1)	$a(b + 1) = (b + 1)a$	$\forall_x \forall_y xy = yx$	UI
	<hr/>		
	$\forall_x \forall_y \forall_z (x + y)z = xz + yz$		
2)	$(b + 1)a = ba + 1 \cdot a$		UI
	<hr/>		
3)	$a(b + 1) = ba + 1 \cdot a$		(2), (1); RRE
	<hr/>		
4)	$ba = ab$	$\forall_x \forall_y xy = yx$	UI
	<hr/>		
5)	$a(b + 1) = ab + 1 \cdot a$		(4), (3); RRE
	<hr/>		
6)	$1 \cdot a = a \cdot 1$	$\forall_x \forall_y xy = yx$	UI
	<hr/>		
7)	$a \cdot 1 = a$	$\forall_x x \cdot 1 = x$	UI
	<hr/>		
8)	$1 \cdot a = a$		(7), (6); RRE
	<hr/>		
9)	$a(b + 1) = ab + a$		(8), (5); RRE
	<hr/>		
10)	$\forall_x \forall_y x(y + 1) = xy + x$		(1) - (9); TPP

We have checked our column derivation and it satisfies conditions (1) and (2), [Page 116]. Hence, it is a valid derivation.

The premisses of our valid derivation are principles:

$$\forall_x \forall_y xy = yx$$

$$\forall_x x \cdot 1 = x$$

$$\forall_x \forall_y \forall_z (x + y)z = xz + yz$$

and the conclusion is:

$$\forall_x \forall_y x(y + 1) = xy + x$$

Hence, we have shown that ' $\forall_x \forall_y x(y + 1) = xy + x$ ' is a theorem. We say that we have **proved** that the sentence ' $\forall_x \forall_y x(y + 1) = xy + x$ ' is a theorem. Note well that the purpose of proof is not to convince you or us of the "correctness" of the generalization. In proving that the generalization is a theorem, we are **not** trying to convince you that the generalization is true. [You are probably as sure of this generalization as you are of the generalizations which we took as principles.] We prove that ' $\forall_x \forall_y x(y + 1) = xy + x$ ' is a theorem by showing that it is a logical consequence of our principles.

[RR]

[Page 123]

You may have a plan which is different from ours. If so, expand your plan into a complete column derivation. Then check to see whether your column satisfies conditions (1) and (2). [See Page 116.] If you have a valid derivation and the only premisses are principles, then you have a proof that the conclusion of the derivation is a theorem.

Note that you could have written a derivation different from ours and still have a proof that ' $\forall_x \forall_y x(y + 1) = xy + x$ ' is a theorem. Usually, there is more than one way to derive a sentence from the principles and hence more than one proof that a sentence is a theorem.

On your work sheet, write a complete derivation which proves that ' $\forall_x x(2x - 3) = 2xx - 3x$ ' is a theorem. [You may find it helpful to write a plan first on scratch paper.]

[RR]

[Page 124]

Does your derivation satisfy conditions (1) and (2) [See Page 116]? Are the only premisses principles? If so, you have proved that ' $\forall_x x(2x - 3) = 2xx - 3x$ ' is a theorem. If you have any doubts about your derivation, consult with your teacher.

Here is our proof that ' $\forall_x x(2x - 3) = 2xx - 3x$ ' is a theorem.

- | | | | |
|-----|----------------------------------|--|----------------|
| (1) | $a(2a - 3) = (2a - 3)a$ | $\forall_x \forall_y xy = yx$ | UI |
| (2) | $(2a - 3)a = 2aa - 3a$ | $\forall_x \forall_y \forall_z (x - y)z = xz - yz$ | UI |
| (3) | $a(2a - 3) = 2aa - 3a$ | | (2), (1); RRE |
| (4) | $\forall_x x(2x - 3) = 2xx - 3x$ | | (1) - (3); TPP |

Turn to Page 125.

{RR}

[Page 125]

We have decided that the generalization:

$$(*) \quad \forall_x \quad 1 + 2x = 3x$$

is not a theorem. Hence, we believe that ' $\forall_x \quad 1 + 2x = 3x$ ' cannot be derived from our principles. What leads us to this belief? Remember, we accepted the principles because they seemed to summarize some of the things we know about numbers and operations with numbers. We have also accepted certain rules of reasoning (UI, RRE, and TPP).

Hence, here is the situation.

(A) We accept the principles as true sentences.

(B) We accept the rules of reasoning (UI, RRE, and TPP).

Therefore, (C) we accept as true statements any statements which follow from our principles by application of our rules of reasoning.

Now, if ' $\forall_x \quad 1 + 2x = 3x$ ' could be derived from our principles, we would accept it as true. If we accept the universal generalization, we are agreeing to accept each of its instances. The sentence:

$$1 + 2 \cdot 5 = 3 \cdot 5$$

is an instance of ' $\forall_x \quad 1 + 2x = 3x$ '. Now, ' $1 + 2 \cdot 5 = 3 \cdot 5$ ' is false; $1 + 2 \cdot 5 = 11$, $3 \cdot 5 = 15$, and $11 \neq 15$. Since at least one instance of the generalization (*) is false, we believe that (*) is not a theorem.

We call the true sentence ' $1 + 2 \cdot 5 \neq 3 \cdot 5$ '

a counter-instance of the generalization.

If a universal generalization has a counter-instance, then we believe that the generalization cannot be derived from our principles and we say that such a generalization is not a theorem.

Turn to PAGE 126.

[RR]

[Page 126]

The generalization ' $\forall_x \forall_y x - y = y - x$ ' has many true instances. Here are a few:

$$3 - 3 = 3 - 3$$

$$8 - 8 = 8 - 8$$

$$\frac{1}{2} - \frac{1}{2} = \frac{1}{2} - \frac{1}{2}$$

The generalization ' $\forall_x \forall_y x - y = y - x$ ' also has many counter-instances. Here are a few:

$$5 - 2 \neq 2 - 5$$

$$\frac{1}{2} - 3 \neq 3 - \frac{1}{2}$$

$$3 - 8 \neq 8 - 3$$

So, although ' $\forall_x \forall_y x - y = y - x$ ' has many true instances, it is not a theorem. The reason it is not a theorem is that it has at least one counter-instance.

Indicate on your work sheet which of these sentences are true.

- (a) A theorem has no counter-instances.
- (b) A "non-theorem" has no true instances.
- (c) If a universal generalization has at least one true instance, then it is a theorem.
- (d) If a universal generalization has at least one counter-instance, then it is not a theorem.

[RR]

[Page 127]

- (a) A theorem has no counter-instances.
- (d) If a universal generalization has at least one counter-instance, then it is not a theorem.

* * *

Decide which of the universal generalizations given below are theorems.
 Indicate your decisions by writing the corresponding letter on your work sheet.
 Give a counter-instance of the generalization if you say it is not a theorem.

- (a) $\forall_x \forall_y \forall_z \ x + (y - z) = (x + y) - z$
- (b) $\forall_x \forall_y \forall_z \ x - (y - z) = (x - y) - z$
- (c) $\forall_x \forall_y \forall_z \ x + (y - z) = (y + x) - z$
- (d) $\forall_x \forall_y \ x(y + y) = y(x + x)$
- (e) $\forall_x \forall_y \ 2x + 2y = 2(x + y)$

[RR]

[Page 128]

The theorems are printed in green.

$$(a) \quad \forall_x \forall_y \forall_z \quad x + (y - z) = (x + y) - z$$

$$(b) \quad \forall_x \forall_y \forall_z \quad x - (y - z) = (x - y) - z$$

$$(c) \quad \forall_x \forall_y \forall_z \quad x + (y - z) = (y + x) - z$$

$$(d) \quad \forall_x \forall_y \quad x(y + y) = y(x + x)$$

$$(e) \quad \forall_x \forall_y \quad 2x + 2y = 2(x + y)$$

* * *

Here is a derivation of the generalization ' $\forall_x \forall_y \forall_z \quad x + (y - z) = (x + y) - z$ ':

- | | | | |
|----|---|--|----|
| 1) | $c + (a - b) = c + (a - b)$ | $\forall_x \quad x = x$ | UI |
| 2) | $a - b = a + -b$ | $\forall_x \forall_y \quad x - y = x + -y$ | UI |
| 3) | $c + (a - b) = c + (a + -b)$ | (2), (1); RRE | |
| | $\forall_x \forall_y \forall_z \quad (x + y) + z = x + (y + z)$ | | |
| 4) | $(c + a) + -b = c + (a + -b)$ | UI | |
| 5) | $c + (a - b) = (c + a) + -b$ | (4), (3); RRE | |
| 6) | $(c + a) - b = (c + a) + -b$ | $\forall_x \forall_y \quad x - y = x + -y$ | UI |
| 7) | $c + (a - b) = (c + a) - b$ | (6), (5); RRE | |
| 8) | $\forall_x \forall_y \forall_z \quad x + (y - z) = (x + y) - z$ | (1) - (7); TPP | |

Complete on your work sheet.

(a) The conclusion of the derivation is: _____?

(b) The premisses of the derivation are:

?

(c) The derivation shows that the sentence ' _____ ' is a consequence of:

?

[RR]

[Page 129]

(a) The conclusion of the derivation is: $\forall_x \forall_y \forall_z x + (y - z) = (x + y) - z$

(b) The premisses of the derivation are:

$$\begin{aligned}\forall_x \forall_y x - y &= x + -y \\ \forall_x x &= x \\ \forall_x \forall_y \forall_z (x + y) + z &= x + (y + z)\end{aligned}$$

(c) The derivation shows that the sentence ' $\forall_x \forall_y \forall_z x + (y - z) = (x + y) - z$ ' is a consequence of:

$$\begin{aligned}\forall_x \forall_y x - y &= x + -y \\ \forall_x x &= x \\ \forall_x \forall_y \forall_z (x + y) + z &= x + (y + z)\end{aligned}$$

* * *

Does the derivation prove that ' $\forall_x \forall_y \forall_z x + (y - z) = (x + y) - z$ ' is a theorem? Since the premisses are principles which we have accepted, the derivation is a proof that the conclusion of the derivation, ' $\forall_x \forall_y \forall_z x + (y - z) = (x + y) - z$ ', is a theorem.

On your work sheet, prove that ' $\forall_x \forall_y 2x + 2y = 2(x + y)$ ' is a theorem.

[RR]

[Page 130]

I

- | | | | |
|-----|--|-------------------------------|----------------|
| 1) | $2a + 2b = 2a + 2b$ | $\forall_x x = x$ | UI |
| 2) | $2a = a2$ | $\forall_x \forall_y xy = yx$ | UI |
| 3) | $2a + 2b = a2 + 2b$ | | (2), (1); RRE |
| 4) | $2b = b2$ | $\forall_x \forall_y xy = yx$ | UI |
| 5) | $2a + 2b = a2 + b2$ | | (4), (3); RRE |
| | $\forall_x \forall_y \forall_z (x + y)z = xz + yz$ | | |
| 6) | $(a + b)2 = a2 + b2$ | | UI |
| 7) | $2a + 2b = (a + b)2$ | | (6), (5); RRE |
| 8) | $(a + b)2 = 2(a + b)$ | $\forall_x \forall_y xy = yx$ | UI |
| 9) | $2a + 2b = 2(a + b)$ | | (8), (7); RRE |
| 10) | $\forall_x \forall_y 2x + 2y = 2(x + y)$ | | (1) - (9); TPP |

II

- | | | | |
|-----|--|-------------------------------|----------------|
| 1) | $2(a + b) = (a + b)2$ | $\forall_x \forall_y xy = yx$ | UI |
| | $\forall_x \forall_y \forall_z (x + y)z = xz + yz$ | | |
| 2) | $(a + b)2 = a2 + b2$ | | UI |
| 3) | $2(a + b) = a2 + b2$ | | (2), (1); RRE |
| 4) | $a2 = 2a$ | $\forall_x \forall_y xy = yx$ | UI |
| 5) | $2(a + b) = 2a + b2$ | | (4), (3); RRE |
| 6) | $b2 = 2b$ | $\forall_x \forall_y xy = yx$ | UI |
| 7) | $2(a + b) = 2a + 2b$ | | (6), (5); RRE |
| 8) | $2a + 2b = 2a + 2b$ | $\forall_x x = x$ | UI |
| 9) | $2a + 2b = 2(a + b)$ | | (7), (8); RRE |
| 10) | $\forall_x \forall_y 2x + 2y = 2(x + y)$ | | (1) - (9); TPP |

If your derivation is different from ours, have your teacher check your work.

[RR]

[Page 131]

Here is a partial tree-diagram derivation of the generalization

' $\forall_x \forall_y \forall_z x + (y - z) = (y + x) - z$ ':

$$\begin{array}{c}
 \frac{\forall_x \forall_y x + y = y + x}{?} \quad ? \qquad \frac{\forall_x \forall_y \forall_z x + (y - z) = (x + y) - z}{a + (b - c) = (a + b) - c} ? \\
 \hline
 a + (b - c) = (b + a) - c \quad ? \\
 \hline
 \forall_x \forall_y \forall_z x + (y - z) = (y + x) - z \quad ?
 \end{array}$$

On your work sheet, write a complete tree-diagram derivation of the generalization ' $\forall_x \forall_y \forall_z x + (y - z) = (y + x) - z$ '.

[RR]

[Page 132]

$$\begin{array}{c}
 \frac{\forall x \forall y \quad x + y = y + x}{a + b = b + a} \text{ UI} \qquad \frac{\forall x \forall y \forall z \quad x + (y - z) = (x + y) - z}{a + (b - c) = (a + b) - c} \text{ UI} \\
 \hline
 a + (b - c) = (b + a) - c \qquad \text{RRE} \\
 \hline
 \forall x \forall y \forall z \quad x + (y - z) = (y + x) - z \qquad \text{TPP}
 \end{array}$$

* * *

Is the derivation above a proof that ' $\forall x \forall y \forall z \quad x + (y - z) = (y + x) - z$ ' is a theorem? The answer is 'YES'. The derivation is valid, the conclusion is ' $\forall x \forall y \forall z \quad x + (y - z) = (y + x) - z$ ' and the premisses are generalizations which we have accepted. One premiss ' $\forall x \forall y \quad x + y = y + x$ ' is a principle and the other premiss ' $\forall x \forall y \forall z \quad x + (y - z) = (x + y) - z$ ' is a theorem. Hence, both premisses are generalizations which we have accepted as true.

Turn to PAGE 133.

[RR]

[Page 133]

Suppose we want to show that the sentence $\forall_x \forall_y (x + 1)y = xy + y$ is a theorem. We might start off this way.

$$(1) \quad \forall_x \forall_y \forall_z (x + y)z = xz + yz \quad [\text{dpma}]$$

$$(2) \quad (a + 1)b = ab + 1 \cdot b \quad [(1); \text{UI}]$$

We notice that we have ' $1 \cdot b$ ' and we want ' b '. If we had ' $b \cdot 1$ ' we could use pml and get ' b '. Now, we could use cpm and get ' $1 \cdot b = b \cdot 1$ ' and then use RRE. But, it seems that we can show that a universal generalization of which ' $1 \cdot b = b$ ' is an instance is a theorem. Such a universal generalization is ' $\forall_x 1 \cdot x = x$ '. So, let's use ' $\forall_x 1 \cdot x = x$ ' as a premiss. Here is the complete derivation:

$$\begin{array}{c} \frac{\forall_x 1 \cdot x = x}{1 \cdot b = b} \text{UI} \qquad \frac{\forall_x \forall_y \forall_z (x + y)z = xz + yz}{(a + 1)b = ab + 1 \cdot b} \text{UI} \\ \hline (a + 1)b = ab + b \quad \text{RRE} \\ \hline \forall_x \forall_y (x + 1)y = xy + y \quad \text{TPP} \end{array}$$

Do we have a proof that ' $\forall_x \forall_y (x + 1)y = xy + y$ ' is a theorem? No, because although we feel reasonably sure that ' $\forall_x 1 \cdot x = x$ ' is a theorem, we have not written a proof that it is. So, we would go back and show that ' $\forall_x 1 \cdot x = x$ ' is a theorem and then use it in our derivation.

On your work sheet, prove that ' $\forall_x 1 \cdot x = x$ ' is a theorem.

[RR]

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Here is our proof that ' $\forall_x 1 \cdot x = x$ ' is a theorem.

$$\begin{array}{c}
 \frac{\forall_x x \cdot 1 = x}{a \cdot 1 = a} \text{ UI} \qquad \frac{\forall_x \forall_y xy = yx}{1 \cdot a = a \cdot 1} \text{ UI} \\
 \hline
 1 \cdot a = a \qquad \text{RRE} \\
 \hline
 \forall_x 1 \cdot x = x \qquad \text{TPP}
 \end{array}$$

You may have written a different derivation. If you have written a valid derivation with ' $\forall_x 1 \cdot x = x$ ' as conclusion and your premisses are principles or theorems, you have also proved that ' $\forall_x 1 \cdot x = x$ ' is a theorem.

* * *

Prove that ' $\forall_x \forall_y 2x(y + 3) = 2xy + (2 \cdot 3)x$ ' is a theorem.

[Hint: First prove that (*) ' $\forall_x \forall_y \forall_z x(y + z) = xy + xz$ ' is a theorem and then use (*) as a premiss in your proof that ' $\forall_x \forall_y 2x(y + 3) = 2xy + (2 \cdot 3)x$ ' is a theorem.]

[RR]

[Page 135]

Do you have a valid derivation? Have you used only principles or theorems as premisses? Have your teacher check your work.

* * *

Since we are now able to use previously derived sentences as premisses, it would be a good idea to keep a record of these sentences. Write 'THEOREMS' at the top of a piece of paper and then write these theorems on the paper.

$$(1) \quad \forall_x \quad 1 \cdot x = x$$

$$(2) \quad \forall_x \quad 0 \cdot x = 0$$

$$(3) \quad \forall_x \forall_y \quad (x + y) - y = x$$

$$(4) \quad \forall_x \forall_y \quad (x + 1)y = xy + y$$

$$(5) \quad \forall_x \forall_y \forall_z \quad x(y + z) = xy + xz$$

From now on, whenever we prove a theorem or you prove a theorem, add it to your list.

Turn to PAGE 136.

[RR]

[Page 136]

We have mentioned that writing a proof that a sentence is a theorem is not done to convince you of the truth of the sentence. We hope that you have a good feeling when you can show that a sentence is a logical consequence of our basic principles. But, there is another benefit from proving that a sentence is a theorem.

Suppose some of the students from Zabbranchburg High have made a trip to a planet in another solar system. [You may do this some day.] Imagine that after having been on this planet for a while they learned that

(A) The name of the planet is 'Glox'.

(B) The natives speak a language different from English.

Turn to PAGE 137a.

[Page 137a]

[RR]

Although the language of Glox (Gloxian) is different from English, some of the books used in their schools resemble the ones used in Zabbranchburg. In looking through one of the Gloxian books Pete, a Zabbranchburg student, found a page which looked like this:

AXIOMS

$$(A_1) \quad \forall_x \forall_y \quad x \dot{+} y = y \dot{+} x$$

$$(A_2) \quad \forall_x \forall_y \quad x \cdot y = y \cdot x$$

$$(A_3) \quad \forall_x \forall_y \forall_z \quad (x \dot{+} y) \dot{+} z = x \dot{+} (y \dot{+} z)$$

$$(A_4) \quad \forall_x \forall_y \quad (x \cdot y) \cdot z = x \cdot (y \cdot z)$$

$$(A_5) \quad \forall_x \forall_y \forall_z \quad (x \dot{+} y) \cdot z = (x \cdot z) \dot{+} (y \cdot z)$$

$$(A_6) \quad \forall_x \quad x \dot{+} 0 = x$$

$$(A_7) \quad \forall_x \quad x \cdot 1 = x$$

$$(A_8) \quad \forall_x \quad x \dot{+} -x = 0$$

$$(A_9) \quad \forall_x \forall_y \quad x - y = x \dot{+} -y$$

$$(A_{10}) \quad \forall_x \forall_y \neq 0 \quad \frac{x}{y} \cdot y = x$$

Although he could read very little else in the book, he saw that some of these AXIOMS resemble some of our principles. So, he found a Gloxian native and wrote:

$$\forall_x \forall_y \quad x \dot{+} y = y \dot{+} x$$

$$2 \dot{+} 3 = 3 \dot{+} 2$$

$$8 \dot{+} 10 = 10 \dot{+} 8$$

$$13 \dot{+} 7 = 7 \dot{+} 13$$

The Gloxian nodded his head and smiled. Pete had already determined that a Gloxian indicates approval by nodding his head and smiling. He indicates disapproval by shaking his head and frowning. Since he smiled this time, Pete was encouraged. After writing several more things like:

$$\forall_x \quad x \dot{+} 0 = x$$

$$2 \dot{+} 0 = 2$$

$$5 \dot{+} 0 = 5$$

[RR]

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$$\begin{aligned}\forall_x \forall_y x \cdot y &= y \cdot x \\ 3 \cdot 5 &= 5 \cdot 3 \\ 8 \cdot 19 &= 19 \cdot 8\end{aligned}$$

and getting a smile each time, Pete decided that Gloxians have a rule of reasoning like the one we call 'Universal Instantiation'. They probably do not use this name, but they do reason the way we do from a sentence which starts out with ' \forall '.

In a similar way, Pete found out that Gloxians have rules like the replacement rule for equations, the law of identity, and the test-pattern principle.

After learning this, Pete wrote

$$\forall_x 1 \cdot x = x$$

The Gloxian nodded his head and smiled. Of course, Pete was sure that the Gloxian would accept the sentence ' $\forall_x 1 \cdot x = x$ '. Why was he sure? Recall that we showed that the sentence ' $\forall_x 1 \cdot x = x$ ' is a theorem. That is, we showed that it is a logical consequence of the principles ' $\forall_x x \cdot 1 = x$ ' and ' $\forall_x xy = yx$ '. Since the Gloxians use UI, RRE, and TPP the same way we do, the sentence ' $\forall_x 1 \cdot x = x$ ' follows from the AXIOMS ' $\forall_x x \cdot 1 = x$ ' and ' $\forall_x x \cdot y = y \cdot x$ '.

Here is another one of our theorems:

$$\forall_x \forall_y 2x(y + 3) = 2xy + (2 \cdot 3)x$$

The corresponding Gloxian sentence is:

$$\forall_x \forall_y 2 \cdot x \cdot (y \dot{+} 3) = [2 \cdot x \cdot y] \dot{+} [(2 \cdot 3) \cdot x]$$

For each of the theorems listed on your theorem sheet, write on your work sheet the corresponding Gloxian sentence. Note that the Gloxians use a ' \cdot ' in places where we would simply place the letters beside one another. [' $2 \cdot x \cdot y$ ' in place of ' $2xy$ '.]

[RR]

[Page 138]

You can be sure that the Gloxians accept each of the sentences that you wrote on your work sheet. We could prove that $\forall_x \forall_y (x \neq 1) \cdot y = (x \cdot y) \neq y$ is a Gloxian theorem by using the derivation on Page 133. Simply put ' \cdot 's in the appropriate places and replace each '+' by a ' \neq '. We would then have a derivation which a Gloxian student might have written. The important point is that without writing a derivation we know that each of the sentences is a Gloxian theorem. The Gloxian number system satisfies our principles [when '+' is interpreted as ' \neq ' and ' \cdot ' is interpreted as ' \cdot '], and hence, each Gloxian sentence which corresponds with one of our theorems is a logical consequence of the Gloxian AXIOMS.

Recall that our principles were written as a summary of some of the things we know about real numbers and operations with real numbers. However, when we derived other sentences from these principles it was only the form of the principles that was important and not the meanings of the principles. This is one of the principle advantages of the deductive method. Since theorems are derived independently of the meaning of the symbols used in the principles and the theorems, these theorems are true sentences about any subject matter which "satisfies" the principles. Any subject matter satisfies the principles if these principles are true sentences about the subject matter when the symbols used in the principles are interpreted as naming operations and objects in the subject matter.

Thus, a deductive treatment of one subject matter provides us with a great deal of information about a second subject provided only that the second subject matter has principles of the same form as the first. You will learn more about this advantage of the deductive method as you continue your study of mathematics.

Turn to PAGE 139.

[RR]

[Page 139]

Encouraged by Pete's results, some of the Zabbranchburg group decided to visit some mathematics classes in a Gloxian school. They were invited to participate in the classes and although they did not understand the Gloxian language they got along quite well. Here is a copy of the first set of exercises done by Pete. His answers are given in script.

$$(a) \ 2 \oplus 3 = 5$$

$$(b) \ 2.5 \oplus 3.5 = 6$$

$$(c) \ 17 \oplus 0 = 17$$

$$(d) \ 4 \cdot 9 = 36$$

$$(e) \ 2 \cdot 3 = 6$$

$$(f) \ 4 \cdot (8 \oplus 5) = 52$$

The Gloxian teacher smiled and wrote 'OK' on Pete's paper.

* * *

Given below is the set of exercises assigned for homework. Pete tried the first three exercises and received Gloxian approval. He then left the class and finished the assignment.

On your work sheet, finish the assignment.

$$(a) \ 3 \oplus 2 = \underline{5}$$

$$(b) \ 6 \oplus 0 = \underline{6}$$

$$(c) \ 10 - 8 = \underline{2}$$

$$(d) \ 12 - 7 = \underline{?}$$

$$(e) \ 36 \oplus 9 = \underline{?}$$

$$(f) \ 14 - 6 = \underline{?}$$

$$(g) \ 8 \cdot 7 = \underline{?}$$

$$(h) \ 9 \cdot 3 = \underline{?}$$

$$(i) \ 5 \oplus (9 \cdot 10) = \underline{?}$$

$$(j) \ 5 \cdot (9 \oplus 10) = \underline{?}$$

$$(k) \ 9 - 3 = \underline{?}$$

$$(l) \ 3 - 9 = \underline{?}$$

$$(m) \ 17 - 26 = \underline{?}$$

$$(n) \ 8 - 0 = \underline{?}$$

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[RR]

$$(a) \quad 3 \div 2 = 5$$

$$(c) \quad 10 - 8 = 2$$

$$(e) \quad 36 \div 9 = 45$$

$$(g) \quad 8 \cdot 7 = 56$$

$$(i) \quad 5 \div (9 \cdot 10) = 95$$

$$(k) \quad 9 - 3 = 6$$

$$(m) \quad 17 - 26 = -9$$

$$(b) \quad 6 \div 0 = 6$$

$$(d) \quad 12 - 7 = 5$$

$$(f) \quad 14 - 6 = 8$$

$$(h) \quad 9 \cdot 3 = 27$$

$$(j) \quad 5 \cdot (9 \div 10) = 95$$

$$(l) \quad 3 - 9 = -6$$

$$(n) \quad 8 - 0 = 8$$

* * *

Pete handed in his paper and the teacher graded it. But this time there was no 'OK' on the paper. Here is what Pete's paper looked like after the teacher graded it.

$$(a) \quad 3 \div 2 = 5 C$$

$$(c) \quad 10 - 8 = 2 C$$

$$(e) \quad 36 \div 9 = 45 C$$

$$(g) \quad 8 \cdot 7 = 56 C$$

$$(i) \quad 5 \div (9 \cdot 10) = 95 C$$

$$(k) \quad 9 - 3 = 6 C$$

$$(m) \quad 17 - 26 = -9 ?$$

$$(b) \quad 6 \div 0 = 6 C$$

$$(d) \quad 12 - 7 = 5 C$$

$$(f) \quad 14 - 6 = 8 C$$

$$(h) \quad 9 \cdot 3 = 27 C$$

$$(j) \quad 5 \cdot (9 \div 10) = 95 C$$

$$(l) \quad 3 - 9 = -6 ?$$

$$(n) \quad 8 - 0 = 8 C$$

[RR]

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[Read the facing page first.]

Pete was puzzled by the '?'s for exercises (l) and (m). But, since he did not understand Gloxian he decided to observe the class again and see if he could figure out what was wrong with his paper.

The teacher wrote these exercises on the board:

$$(a) \quad 5 \oplus {}^n3 = ?$$

$$(b) \quad 8 \oplus {}^n6 = ?$$

$$(c) \quad 9 \oplus {}^n5 = ?$$

$$(d) \quad {}^n3 \oplus -2 = ?$$

$$(e) \quad 8 \oplus -6 = ?$$

$$(f) \quad 5 \oplus -3 = ?$$

and a Gloxian student did them correctly. Here is how he completed them.

$$(a) \quad 5 \oplus {}^n3 = 2$$

$$(b) \quad 8 \oplus {}^n6 = 2$$

$$(c) \quad 9 \oplus {}^n5 = 4$$

$$(d) \quad {}^n3 \oplus -2 = {}^n5$$

$$(e) \quad 8 \oplus -6 = 2$$

$$(f) \quad 5 \oplus -3 = 2$$

After seeing this, Pete changed his answers for (l) and (m). In place of '-6' he wrote 'ⁿ6' and in place of '-9' he wrote 'ⁿ9'. The teacher then marked 'OK' on his paper. That night Pete did the class assignment and received an 'OK'.

Here is the assignment. Do it on your work sheet. Use Gloxian numerals.

$$(a) \quad 3 \oplus {}^n2 = ?$$

$$(b) \quad {}^n12 \oplus {}^n13 = ?$$

$$(c) \quad 4 - 17 = ?$$

$$(d) \quad 5 - {}^n2 = ?$$

$$(e) \quad 12 - {}^n10 = ?$$

$$(f) \quad {}^n4 - {}^n3 = ?$$

[RR]

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- (a) 1 (b) n25 (c) n13 (d) 7 (e) 22 (f) n1

* * *

Some of Pete's earth group were quite impressed with his ability to do Gloxian mathematics. Pete explained it this way:

In class the other day I noticed that $5 \oplus {}^n3 = 2$ and $5 \oplus -3 = 2$

It follows that ${}^n3 = -3$. I also figured out that ${}^n6 = -6$.

So, I did the first homework problem this way. Since

${}^n6 = -6$ and ${}^n3 = -3$, I assumed that ${}^n2 = -2$. So,

$3 \oplus {}^n2 = 3 \oplus -2$.

Ann asked, "But how do you know that ${}^n2 = -2$?"

Pete said, "I don't, but if ${}^n2 = -2$ then $3 + {}^n2 = 3 + -2$."

Did Pete reason correctly? Let's write a column derivation of the sentence ' $3 \oplus {}^n2 = 3 \oplus -2$ '.

(1)	${}^n2 = -2$	Assumption	
(2)	$3 \oplus {}^n2 = 3 \oplus {}^n2$	$\forall_x x = x$	UI
(3)	$3 \oplus {}^n2 = 3 \oplus -2$		(1), (2); RRE

Does this derivation show that ' $3 \oplus {}^n2 = 1$ ' is a theorem? No, because at least one premiss [${}^n2 = -2$] is not a Gloxian axiom, Gloxian theorem, or justified by the Law of Identity. Although we have not shown that ' $3 + {}^n2 = 3 \oplus -2$ ' is a theorem, we have shown that the sentence 'if ${}^n2 = -2$ then $3 \oplus {}^n2 = 3 \oplus -2$ ' is a consequence of just ' $\forall_x x = x$ ', Hence, the sentence:

'if ${}^n2 = -2$ then $3 \oplus {}^n2 = 3 \oplus -2$ '

is a theorem.

Show that the sentence ' $9 \oplus {}^n3 = 9 \oplus -3$ ' follows from the sentences ' ${}^n3 = -3$ ' and ' $\forall_x x = x$ '. Write a derivation on your work sheet.

[RR]

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- | | | | |
|-----|---|-------------------|---------------|
| (1) | $n_3 = -3$ | <u>Assumption</u> | |
| (2) | $9 \nleftrightarrow n_3 = 9 \nleftrightarrow n_3$ | $\forall_x x = x$ | UI |
| (3) | $9 \nleftrightarrow n_3 = 9 \nleftrightarrow -3$ | | (1), (2); RRE |

* * *

Ann would accept the sentence:

$$3 \nleftrightarrow n_2 = 3 \nleftrightarrow -2$$

with the condition that $n_2 = -2$. So, she would accept the conditional sentence:

$$\text{if } n_2 = -2 \text{ then } 3 \nleftrightarrow n_2 = 3 \nleftrightarrow -2.$$

The "if-part":

$$n_2 = -2$$

is called the antecedent of the conditional and the "then-part":

$$3 \nleftrightarrow n_2 = 3 \nleftrightarrow -2$$

is called the consequent of the conditional [Look up antecedent and consequent in a dictionary.]

Complete these sentences on your work sheet.

- (a) The sentence ' $n_8 = -8$ ' is the _____ of the condition ' $\text{if } n_8 = -8 \text{ then } 13 \nleftrightarrow n_8 = 13 \nleftrightarrow -8$ '.
- (b) The sentence ' $13 \nleftrightarrow n_8 = 13 \nleftrightarrow -8$ ' is the _____ of the conditional ' $\text{if } n_8 = -8 \text{ then } 13 \nleftrightarrow n_8 = 13 \nleftrightarrow -8$ '.

[RR]

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- (a) The sentence ' $n8 = -8$ ' is the antecedent of the conditional ' $\text{if } n8 = -8 \text{ then } 13 \nabla n8 = 13 \nabla -8$ '.
- (b) The sentence ' $13 \nabla n8 = 13 \nabla -8$ ' is the consequent of the conditional ' $\text{if } n8 = -8 \text{ then } 13 \nabla n8 = 13 \nabla -8$ '.

* * *

For each of these conditional sentences, tell its antecedent and its consequent.

- (a) If $3 = 3$ then $5 + 3 = 5 + 3$.
- (b) If $a = b$ then $a + c = b + c$.
- (c) If bats are birds then bats lay eggs.
- (d) If Jeff lives in Ames then Jeff lives in Iowa.

[RR]

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	<u>antecedent</u>	<u>consequent</u>
(a) If $3 = 3$ then $5 + 3 = 5 + 3$.	$3 = 3$	$5 + 3 = 5 + 3$
(b) If $a = b$ then $a + c = b + c$.	$a = b$	$a + c = b + c$
(c) If bats are birds then bats lay eggs.	bats are birds	bats lay eggs
(d) If Jeff lives in Ames then Jeff lives in Iowa.	Jeff lives in Ames	Jeff lives in Iowa

* * *

Here are four more conditional sentences.

- (1) If $A \in \overline{BC}$, then $BA + AC = BC$.
- (2) If $a = b$ then $ac = bc$.
- (3) If $a + 1 = 2$ then $a = 1$.
- (4) If $a \neq b$ then $a > b$ or $b > a$.

Complete these sentences on your work sheet.

- (a) ' $A \in \overline{BC}$ ' is the _____ of sentence (1).
- (b) _____ is the consequent of sentence (2).
- (c) ' $a = 1$ ' is the _____ of sentence (3).
- (d) ' $a \neq b$ ' is the _____ of sentence (4).
- (e) ' $a > b$ or $b > a$ ' is the _____ of sentence (4).
- (f) _____ is the antecedent of sentence (3).

[RR]

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- (a) ' $A \in \overline{BC}$ ' is the antecedent of sentence (1).
- (b) $ac = bc$ is the consequent of sentence (2).
- (c) ' $a = 1$ ' is the consequent of sentence (3).
- (d) ' $a \neq b$ ' is the antecedent of sentence (4).
- (e) ' $a > b$ or $b > a$ ' is the consequent of sentence (4).
- (f) $a + 1 = 2$ is the antecedent of sentence (3).

Turn to PAGE 146.

[RR]

[Page 146]

Ann agreed to accept the conditional sentence:

$$\text{if } n_2 = -2 \text{ then } 3 \oplus n_2 = 3 \oplus -2$$

But she still objected to accepting the sentence ' $3 \oplus n_2 = 3 \oplus -2$ '.

Pete replied, "I also found out from one of my Gloxian friends that $n_2 = -2$.

So, from the conditional sentence:

$$\text{if } n_2 = -2 \text{ then } 3 \oplus n_2 = 3 \oplus -2$$

and its antecedent:

$$n_2 = -2$$

I concluded that $3 \oplus n_2 = 3 \oplus -2$."

The inference:

$$(\star) \quad \frac{n_2 = -2 \quad \text{if } n_2 = -2 \text{ then } 3 \oplus n_2 = 3 \oplus -2}{3 \oplus n_2 = 3 \oplus -2}$$

shows the reasoning Pete used.

Complete these sentences on your work sheet.

(a) The conclusion of the inference (\star) is: _____ ?

(b) The premisses of the inference (\star) are: _____ ?

[RR]

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(a) The conclusion of the inference (☆) is: $3 \oplus n_2 = 3 \oplus -2$

(b) The premisses of the inference (☆) are:

$$\text{if } n_2 = -2 \text{ then } 3 \oplus n_2 = 3 \oplus -2$$

$$n_2 = -2$$

* * *

Each of the following exercises contains two premisses and a conclusion. Your job is to decide in each case if the conclusion follows logically from the premisses. For each exercise, if you think the conclusion does logically follow from the premisses, write 'YES'; if you think the conclusion does not logically follow from the premisses, write 'NO'.

(a)
$$\frac{3 = 3 \quad \text{if } 3 = 3 \text{ then } 5 + 3 = 5 + 3}{5 + 3 = 5 + 3}$$

(b)
$$\frac{a = b \quad \text{if } a = b \text{ then } a + c = b + c}{a + c = b + c}$$

(c)
$$\frac{\text{bats are birds} \quad \text{if bats are birds then bats lay eggs}}{\text{bats lay eggs}}$$

[RR]

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Yes (a) $\frac{3 = 3}{5 + 3 = 5 + 3}$ if $3 = 3$ then $5 + 3 = 5 + 3$

Yes (b) $\frac{a = b}{a + c = b + c}$ if $a = b$ then $a + c = b + c$

Yes (c) $\frac{\text{bats are birds}}{\text{bats lay eggs}}$ if bats are birds then bats lay eggs

[Even though the conclusion 'bats lay eggs' is false, it logically follows from the premisses.]

* * *

For each inference in which you think the conclusion logically follows from the premisses, write 'YES'. Otherwise, write 'NO'.

(a) $\frac{\text{It is raining}}{\text{The streets are wet}}$ If it is raining then the streets are wet

(b) $\frac{\text{The streets are wet}}{\text{It is raining}}$ If it is raining then the streets are wet

(c) $\frac{\text{John is a good basketball player}}{\text{John is tall}}$ If John is tall then John is a good basketball player

(d) $\frac{\text{John is tall}}{\text{John is a good basketball player}}$ If John is tall then John is a good basketball player

Turn to PAGE 149a.

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[RR]

- Yes (a) It is raining If it is raining then
the streets are wet
The streets are wet
- No (b) The streets are wet If it is raining then
the streets are wet
It is raining
- No (c) John is a good If John is tall then John
basketball player is a good basketball player
John is tall
- Yes (d) John is tall If John is tall then John is
a good basketball player
John is a good basketball player

SKIP THIS PAGE

[RR]

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For each inference in which you think the conclusion logically follows from the premisses, write 'YES'. Otherwise, write 'NO'. [Note that you are not being asked to decide whether the premisses or conclusions are true.]

$$(a) \frac{\text{Jeff lives in Iowa} \quad \text{If Jeff lives in Ames then Jeff lives in Iowa}}{\text{Jeff lives in Ames}}$$

$$(b) \frac{A \in \overline{BC} \quad \text{if } A \in \overline{BC} \text{ then } BA + AC = BC}{BA + AC = BC}$$

$$(c) \frac{ac = bc \quad \text{if } a = b \text{ then } ac = bc}{a = b}$$

$$(d) \frac{a = 1 \quad \text{if } a + 1 = 2 \text{ then } a = 1}{a + 1 = 2}$$

$$(e) \frac{2 = 3 \quad \text{if } 2 = 3 \text{ then } 2 + 5 = 3 + 5}{2 + 5 = 3 + 5}$$

$$(f) \frac{a \neq b \quad \text{if } a \neq b \text{ then } a > b \text{ or } b > a}{a > b \text{ or } b > a}$$

Turn to PAGE 150a.

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[RR]

No (a)
$$\frac{\text{Jeff lives in Iowa} \quad \text{If Jeff lives in Ames then Jeff lives in Iowa}}{\text{Jeff lives in Ames}}$$

Yes (b)
$$\frac{A \in \overline{BC} \quad \text{if } A \in \overline{BC} \text{ then } BA + AC = BC}{BA + AC = BC}$$

No (c)
$$\frac{ac = bc \quad \text{if } a = b \text{ then } ac = bc}{a = b}$$

No (d)
$$\frac{a = 1 \quad \text{if } a + 1 = 2 \text{ then } a = 1}{a + 1 = 2}$$

Yes (e)
$$\frac{2 = 3 \quad \text{if } 2 = 3 \text{ then } 2 + 5 = 3 + 5}{2 + 5 = 3 + 5}$$

Yes (f)
$$\frac{a \neq b \quad \text{if } a \neq b \text{ then } a > b \text{ or } b > a}{a > b \text{ or } b > a}$$

SKIP THIS PAGE

[RR]

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The rule of reasoning which justifies inferences (b), (e), and (f) is called modus ponendo ponens, or more often, simply modus ponens. [Find out the translation of the Latin phrase 'modus ponendo ponens'.] Modus ponens tells you that from an if-then sentence:

$$\text{if } A \in \overline{BC} \text{ then } BA + AC = BC$$

and its antecedent:

$$A \in \overline{BC}$$

you may infer its consequent:

$$BA + AC = BC$$

Check that each of the inferences for which we wrote 'YES' is an example of reasoning justified by modus ponens.

Each of the following exercises refers to an example of inferring a conclusion from two premisses by means of modus ponens. Your job is to fill in the missing sentence.

$$(a) \frac{a = b \quad \text{if } a = b \text{ then } c = d}{?}$$

$$(b) \frac{? \quad \text{if } A \in \overline{BC} \text{ then } A \in \overline{BC}}{A \in \overline{BC}}$$

$$(c) \frac{a + b = 0 \quad ?}{b = -a}$$

$$(d) \frac{? \quad \text{if } ab = c \text{ then } b = c \div a}{b = c \div a}$$

[RR]

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- (a) $\frac{a = b}{c = d} \quad \frac{\text{if } a = b \text{ then } c = d}{c = d}$
- (b) $\frac{A \in \overline{BC}}{A \in \overline{BC}} \quad \frac{\text{if } A \in \overline{BC} \text{ then } A \in \overline{BC}}{A \in \overline{BC}}$
- (c) $\frac{a + b = 0}{b = -a} \quad \frac{\text{if } a + b = 0 \text{ then } b = -a}{b = -a}$
- (d) $\frac{ab = c}{b = c \div a} \quad \frac{\text{if } ab = c \text{ then } b = c \div a}{b = c \div a}$

[You could probably do (b) correctly even if you do not know what ' $A \in \overline{BC}$ ' means.]

* * *

Complete on your work sheet.

	Antecedent	Consequent
(a) if $a = b$ then $c = d$	<u> ?</u>	<u> ?</u>
(b) if $A \in \overline{BC}$ then $A \in \overline{BC}$	<u> ?</u>	<u> ?</u>
(c) if $a + b = 0$ then $b = -a$	<u> ?</u>	<u> ?</u>
(d) if $ab = c$ then $b = c \div a$	<u> ?</u>	<u> ?</u>

[RR]

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	Antecedent	Consequent
(a) if $a = b$ then $c = d$	$a = b$	$c = d$
(b) if $A \in \overline{BC}$ then $A \in \overline{BC}$	$A \in \overline{BC}$	$A \in \overline{BC}$
(c) if $a + b = 0$ then $b = -a$	$a + b = 0$	$b = -a$
(d) if $ab = c$ then $b = c \div a$	$ab = c$	$b = c \div a$

* * *

Fill in the missing sentences so that each inference is one which is justified by modus ponens.

$$(a) \frac{a - b \neq 0 \quad ?}{a \neq b}$$

$$(b) \frac{A \in \ell \quad \text{if } A \in \ell \text{ then } A \in n}{?}$$

$$(c) \frac{A \notin \ell \quad ?}{A \notin n}$$

$$(d) \frac{? \quad \text{if } \ell \parallel n \text{ then } \ell \cap n = \emptyset}{\ell \cap n = \emptyset}$$

Turn to PAGE 153a.

[Page 153a]

[RR]

$$(a) \frac{a - b \neq 0 \quad \text{if } a - b \neq 0 \text{ then } a \neq b}{a \neq b}$$

$$(b) \frac{A \in \ell \quad \text{if } A \in \ell \text{ then } A \in n}{A \in n}$$

$$(c) \frac{A \notin \ell \quad \text{if } A \notin \ell \text{ then } A \notin n}{A \notin n}$$

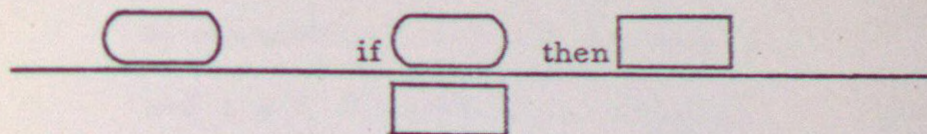
$$(d) \frac{\ell \parallel n \quad \text{if } \ell \parallel n \text{ then } \ell \cap n = \emptyset}{\ell \cap n = \emptyset}$$

SKIP THIS PAGE

[RR]

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Each of the inferences (a)-(d) could be obtained from the pattern:



For example, inference (a) could be obtained by writing 'a - b ≠ 0' in each ' $\boxed{}$ ' and 'a ≠ b' in each ' $\boxed{}$ '.

Hence, modus ponens, the rule which justifies each of the inferences (a)-(d), may be stated as:

The result of substituting sentences for 'p' and 'q' in:

$$* \frac{p \quad \text{if } p \text{ then } q}{q}$$

is a valid inference.

Since $\frac{p \quad \text{if } p \text{ then } q}{q}$ is a pattern for writing inferences, we will call

* an inference pattern

Complete on your work sheet.

(a) Inference (a) may be obtained from the inference pattern:

$$* \frac{p \quad \text{if } p \text{ then } q}{q}$$

by substituting ? for each 'p'
and ? for each 'q'.

(b) Inference (b) may be obtained from * by substituting

 ? for each 'p' and
 ? for each 'q'.

[RR]

[Page 154]

- (a) Inference (a) may be obtained from *
by substituting ' $a - b \neq 0$ ' for each ' p '
and ' $a \neq b$ ' for each ' q '.
- (b) Inference (b) may be obtained from * by
substituting ' $A \in l$ ' for each ' p ' and
' $A \in n$ ' for each ' q '.

* * *

We can state our new rule of inference more briefly by

MODUS PONENS		
p	if p then q	
q		
is a valid inference pattern.		

Here is an example of how we can use our new rule of reasoning ['MP' for 'modus ponens'] together with RRE in a derivation.

$$\begin{array}{rcl}
 & a \neq b & \text{if } a \neq b \text{ then } c \neq d \\
 d = e & \frac{\quad}{c \neq d} & \text{MP} \\
 & \frac{\quad}{c \neq e} & \text{RRE}
 \end{array}$$

Each of the following exercises gives a pattern for a sequence of inferences. On your work sheet, write the complete sequence and give the rule of reasoning which justifies each inference.

(a)

$$\begin{array}{rcl}
 a = b & a \neq c & ? \\
 \frac{\quad}{b \neq c} & & ? \\
 \frac{\quad}{b \neq d} & & \text{MP}
 \end{array}$$

(b)

$$\begin{array}{rcl}
 & ? & \text{if } l \parallel m \text{ then } m \parallel n \\
 k = n & \frac{\quad}{m \parallel n} & ? \\
 \frac{\quad}{?} & & ?
 \end{array}$$

[RR]

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(a)

$$\frac{\frac{a = b \quad a \neq c}{b \neq c} \quad \text{RRE} \quad \text{if } b \neq c \text{ then } b \neq d}{b \neq d} \text{MP}$$

(b)

$$\frac{\frac{k = n \quad \frac{\ell \parallel m \quad \text{if } \ell \parallel m \text{ then } m \parallel n}{m \parallel n} \text{MP}}{m \parallel k} \text{RRE}}$$

* * *

Let's now return to Pete and his Zabbranchburg classmates on Glox. Pete had learned from his Gloxian friend that ${}^n2 = -2$. He wanted to use the sentence ' ${}^n2 = -2$ ' and derive the sentence ' $3 \nabla {}^n2 = 3 \nabla -2$ '. What other premiss did he need in order to use modus ponens?

Complete this inference on your work sheet.

$$\frac{{}^n2 = -2 \quad ?}{3 \nabla {}^n2 = 3 \nabla -2}$$

Turn to PAGE 156a.

[Page 156a]

[RR]

$$\frac{n_2 = -2 \quad \text{if } n_2 = -2 \text{ then } 3 \oplus n_2 = 3 \oplus -2}{3 \oplus n_2 = 3 \oplus -2}$$

* * *

Pete could do lots of Gloxian arithmetic problems like:

- (i) $5 \oplus n_7 = ?$ (ii) $13 \oplus n_{16} = ?$
 (iii) $48 \oplus n_{17} = ?$ (iv) $25 \oplus -3 = ?$

In order to do the problem ' $5 \oplus n_7 = ?$ ', he found out from a Gloxian friend that $n_7 = -7$. He then used the sentence 'if $n_7 = -7$ then $5 \oplus n_7 = 5 \oplus -7$ ' and derived the sentence ' $5 \oplus n_7 = 5 \oplus -7$ '

His inference looked like this:

$$\frac{n_7 = -7 \quad \text{if } n_7 = -7 \text{ then } 5 \oplus n_7 = 5 \oplus -7}{5 \oplus n_7 = 5 \oplus -7}$$

Since he also knew that $5 \oplus -7 = -2$, he completed the problem this way:

$$\frac{\begin{array}{l} n_7 = -7 \\ 5 \oplus -7 = -2 \end{array} \quad \frac{\text{if } n_7 = -7 \text{ then } 5 \oplus n_7 = 5 \oplus -7}{5 \oplus n_7 = 5 \oplus -7}}{5 \oplus n_7 = -2} \quad \begin{array}{l} \text{MP} \\ \text{RRE} \end{array}$$

Ann objected again. "But where did you get the sentence 'if $n_7 = -7$ then $5 \oplus n_7 = 5 \oplus -7$ '?"

SKIP THIS PAGE

[RR]

[Page 156]

Pete replied, "Look, I can derive the sentence 'if $\neg 7 = -7$ then $5 \div \neg 7 = 5 \div -7$ '. Here is my derivation:

- | | | | |
|-----|--|-------------------|---------------|
| (1) | $\neg 7 = -7$ | Assumption | |
| (2) | $5 \div \neg 7 = 5 \div \neg 7$ | $\forall_x x = x$ | UI |
| (3) | $5 \div \neg 7 = 5 \div -7$ | | (1), (2); RRE |
| (4) | <u>if $\neg 7 = -7$ then $5 \div \neg 7 = 5 \div -7$</u> | | |

But Pete couldn't justify his last step. Bill helped him out. Bill said, "I remember doing things like that in algebra. We called it conditionalizing. A conditional sentence can be inferred from its consequent."

Pete continued, "Oh, I remember now. Ann, you accept this sentence:

This planet is Glox.

so you ought to accept this sentence:

If this planet is green, then this planet is Glox.

In fact, you ought to accept this sentence:

If the moon is made of green cheese, then this planet is Glox."

Ann replied, "OK. I guess if I accept the sentence 'This planet is Glox' without any conditions, I have to accept any conditional sentence which has 'this planet is Glox' as a consequent. It seems strange, but I can't see where it will get me into any trouble."

Turn to PAGE 157.

[RR]

[Page 157]

Let's look at Pete's derivation again.

(1)	$\neg 7 = \neg 7$	Assumption	
(2)	$5 \vdash \neg 7 = 5 \vdash \neg 7$	$\forall_x x = x$	UI
(3)	$5 \vdash \neg 7 = 5 \vdash \neg 7$		(1), (2); RRE
(4)	if $\neg 7 = \neg 7$ then $5 \vdash \neg 7 = 5 \vdash \neg 7$		Conditionalizing (3) Discharging (1)

The comments for step (4) refer to two new rules of reasoning. The first of these--conditionalizing--says that a conditional sentence may be inferred from its consequent. In this case:

$$(I) \quad \frac{(3) \quad 5 \vdash \neg 7 = 5 \vdash \neg 7}{(4) \quad \text{if } \neg 7 = \neg 7 \text{ then } 5 \vdash \neg 7 = 5 \vdash \neg 7}$$

Here is a pattern for conditionalizing:

$$(II) \quad \frac{q}{\text{if } p \text{ then } q}$$

On your work sheet, complete this sentence:

The inference (I) may be obtained from the inference pattern (II) by substituting _____ for each 'q' and _____ for 'p'.

[RR]

[Page 158]

$$q: 5 \nabla n_7 = 5 \nabla -7$$

$$p: n_7 = -7$$

* * *

The second new rule---the discharge rule---says that if you derive the consequent:

$$5 \nabla n_7 = 5 \nabla -7$$

of a conditional:

$$\text{if } n_7 = -7 \text{ then } 5 \nabla n_7 = 5 \nabla -7$$

from its antecedent:

$$n_7 = -7$$

and other premisses:

$$\forall_x x = x$$

then the conditional itself:

$$\text{if } n_7 = -7 \text{ then } 5 \nabla n_7 = 5 \nabla -7$$

is a consequence of the other premisses alone.

When we use this rule, we call the antecedent an assumption and indicate that the conditional does not depend on this assumption by saying that the assumption is discharged.

Complete this derivation by using conditionalizing and the discharge rule:

$n_{16} = -16$	$\forall_x x = x$	
	$13 \nabla n_{16} = 13 \nabla n_{16}$	UI
		RRE
$13 \nabla n_{16} = 13 \nabla -16$		C; D *

[RR]

[Page 159]

$$\begin{array}{rcl}
 & \forall x \ x = x & \\
 \frac{n_{16} = -16}{13 \nabla n_{16} = 13 \nabla n_{16}} & & \text{UI} \\
 & & \text{RRE} \\
 \frac{13 \nabla n_{16} = 13 \nabla -16}{\text{if } n_{16} = -16 \text{ then } 13 \nabla n_{16} = 13 \nabla -16} & & \text{C; D}
 \end{array}$$

Note that we use '*'s to show the use of the discharge rule.

* * *

Here is a more concise statement of the discharge rule.

THE DISCHARGE RULE

If the consequent of a conditional sentence is a consequence of the antecedent of the conditional sentence, and other premisses, then the conditional sentence is a consequence of the other premisses alone.

The rule gives you a strategy to use in deriving a conditional sentence from given premisses. Use the antecedent of the conditional as an assumption or an extra premiss. Then, try to derive the consequent of the conditional from this assumption together with the other premisses. When you have done this, then conditionalize the consequent and discharge the extra premiss.

Here is an incomplete derivation of:

$$\text{if } n_8 = -8 \text{ then } 5 \nabla n_8 = 5 \nabla -8$$

- (1) $n_8 = -8$ Assumption
- (2)
- (3)
- (4)

On your work sheet, complete the derivation.

[RR]

[Page 160]

- | | | | |
|-----|--|-----------------------|--------------|
| (1) | $n_8 = -8$ | <u>Assumption - *</u> | |
| (2) | $5 \nabla n_8 = 5 \nabla n_8$ | $\forall_x x = x$ | UI |
| (3) | $5 \nabla n_8 = 5 \nabla -8$ | (1); (2); RRE | |
| (4) | <u>if $n_8 = -8$ then $5 \nabla n_8 = 5 \nabla -8$</u> | | C(3); D(1) * |

* * *

Pete soon got tired of deriving sentences like:

- (i) if $n_8 = -8$ then $5 \nabla n_8 = 5 \nabla -8$
- (ii) if $n_{16} = -16$ then $13 \nabla n_{16} = 13 \nabla -16$
- (iii) if $-34 = n_{34}$ then $18 \nabla -34 = 18 \nabla n_{34}$
- (iv) if $-9 = n_9$ then $37 \nabla -9 = 37 \nabla n_9$

and he noticed that each of these sentences fits the pattern:

$$\text{if } \boxed{} = \bigcirc \quad \text{then } \bigcirc \nabla \boxed{} = \bigcirc \nabla \bigcirc$$

Write on your work sheet a universal generalization of which sentences (i), (ii), (iii), and (iv) are instances.

[RR]

[Page 161]

- (i) if ${}^n8 = -8$ then $5 \oplus {}^n8 = 5 \oplus -8$
 (ii) if ${}^n16 = -16$ then $13 \oplus {}^n16 = 13 \oplus -16$
 (iii) if $-34 = {}^n34$ then $18 \oplus -34 = 18 \oplus {}^n34$
 (iv) if $-9 = {}^n9$ then $37 \oplus -9 = 37 \oplus {}^n9$

are instances of:

$$\forall_x \forall_y \forall_z \text{ if } x = y \text{ then } z \oplus x = z \oplus y$$

* * *

Derive the generalization:

$$\forall_x \forall_y \forall_z \text{ if } x = y \text{ then } z \oplus x = z \oplus y$$

[Hint. Use pattern variables 'a', 'b', and 'c', and the example on Page 159.]

[RR]

[Page 162]

*	$\forall x \ x = x$	
$a = b$	$c \leftrightarrow a = c \leftrightarrow a$	UI
	$c \leftrightarrow a = c \leftrightarrow b$	RRE
	if $a = b$ then $c \leftrightarrow a = c \leftrightarrow b$	C, DR *
	$\forall x \forall y \forall z$ if $x = y$ then $z \leftrightarrow x = z \leftrightarrow y$	TPP

Note that we have used 'C' as an abbreviation for 'conditionalizing', 'DR' as an abbreviation for 'discharge rule', and an asterisk (*) to show the assumption which is discharged.

* * *

Recall that Pete derived the sentence:

$$n_3 = -3$$

from the premisses:

$$5 \leftrightarrow n_3 = 2$$

$$5 \leftrightarrow -3 = 2$$

When questioned about this, Pete wrote:

$5 \leftrightarrow -3 = 2$	$5 \leftrightarrow n_3 = 2$	$\forall x \forall y \forall z$ if $x = y$ then $z \leftrightarrow x = z \leftrightarrow y$
$5 \leftrightarrow n_3 = 5 \leftrightarrow -3$	$n_3 = -3$	if $n_3 = -3$ then $5 \leftrightarrow n_3 = 5 \leftrightarrow -3$

Did Pete write a valid derivation?

[RR]

[Page 163]

Pete's derivation is not valid. One of the inferences:

$$(1) \frac{5 \div n_3 = 5 \div -3 \quad \boxed{\text{N}} \quad \boxed{\text{if } n_3 = -3 \text{ then } 5 \div n_3 = 5 \div -3}}{n_3 = -3}$$

is invalid. Hence, the derivation is invalid. Pete has committed the fallacy of "affirming the consequent".

A modus ponens type inference is of the form

$$\frac{p \quad \text{if } p \text{ then } q}{q}$$

Inference (1) is an example of invalid reasoning

$$\frac{q \quad \boxed{\text{N}} \quad \boxed{\text{if } p \text{ then } q}}{p}$$

Doris wrote this derivation:

$$\frac{5 \div -3 = 2 \quad 5 \div n_3 = 2 \quad \text{if } 5 \div n_3 = 5 \div -3 \text{ then } n_3 = -3}{5 \div n_3 = 5 \div -3 \quad n_3 = -3}$$

On your work sheet, indicate the rule of reasoning which justifies each inference.

[RR]

[Page 164]

$$\begin{array}{l}
 \frac{5 \div -3 = 2 \qquad 5 \div {}^n3 = 2}{5 \div {}^n3 = 5 \div -3} \text{RRE} \\
 \hline
 \frac{\qquad \qquad \qquad {}^n3 = -3}{\text{if } 5 \div {}^n3 = 5 \div -3 \text{ then } {}^n3 = -3} \text{MP}
 \end{array}$$

* * *

Ann objected again. "Where did you get the premiss 'if $5 \div {}^n3 = 5 \div -3$ then ${}^n3 = -3$ '?"

Doris replied, "It's an instance of:

$$\forall_x \forall_y \forall_z \text{ if } x \div y = x \div z \text{ then } y = z."$$

Ann said. "But you haven't shown that ' $\forall_x \forall_y \forall_z \text{ if } x \div y = x \div y \text{ then } y = z$ ' is a theorem."

Doris answered, "Oh, that's not hard to do."

Are you as clever as Doris? In order to prove that ' $\forall_x \forall_y \forall_z \text{ if } x \div y = x \div z \text{ then } y = z$ ' is a theorem, you must write a valid derivation in which

' $\forall_x \forall_y \forall_z \text{ if } x \div y = x \div z \text{ then } y = z$ ' is the conclusion and the only undischarged premisses are AXIOMS, ' $\forall_x x = x$ ', or previously derived sentences.

Before attempting the proof, complete these sentences.

- The generalization ' $\forall_x \forall_y \forall_z \text{ if } x \div y = x \div z \text{ then } y = z$ ' is a logical consequence of the pattern sentence 'if $a \div b = \underline{\quad ? \quad}$ then $\underline{\quad ? \quad}$ '.
- Since the pattern sentence is a conditional sentence, the $\underline{\quad ? \quad}$ of the conditional sentence could be used as a premiss and then discharged.
- AXIOMS, ' $\forall_x x = x$ ', and $\underline{\quad ? \quad}$ can be used as premisses in a proof that a given sentence is a theorem.

[RR]

[Page 165]

- (a) The generalization ' $\forall_x \forall_y \forall_z$ if $x \neq y = x \neq z$ then $y = z$ ' is a logical consequence of the pattern sentence 'if $a \neq b = a \neq c$ then $b = c$ '
- (b) Since the pattern sentence is a conditional sentence, the antecedent of the conditional sentence could be used as a premiss and then discharged.
- (c) AXIOMS, ' $\forall_x x = x$ ', and previously derived theorems can be used as premisses in a proof that a given sentence is a theorem.

* * *

Now, on your work sheet, write a proof that

$\forall_x \forall_y \forall_z$ if $x \neq y = x \neq z$ then $y = z$

is a theorem.

[RR]

[Page 166]

If you were able to write what you think is a proof, have your teacher check your work. Then read the material below.

If you could not write what you think is a proof, read the material given below.

* * *

The pattern sentence we want to derive is:

$$\text{if } a \dot{\vee} b = a \dot{\vee} c \text{ then } b = c.$$

A good way to start a derivation of a conditional is to start with the antecedent of the conditional as a premiss. [We hope to derive the consequent and then conditionalize and discharge.] So we start with:

$$(1) \quad a \dot{\vee} b = a \dot{\vee} c$$

But where do we go from (1)? We note that there are 'a's in the premiss (1) and no 'a's in the desired conclusion ' $b = c$ '. Now, one way to "get rid of" an 'a' is to "add $\neg a$ ". We just proved a theorem which we can use to derive ' $b = c$ ' from ' $a \dot{\vee} b = a \dot{\vee} c$ '. Here is the theorem:

$$(2) \quad \forall_x \forall_y \forall_z \text{ if } x = y \text{ then } z \dot{\vee} x = z \dot{\vee} y$$

and the appropriate instance:

$$(3) \quad \text{if } a \dot{\vee} b = a \dot{\vee} c \text{ then } \neg a \dot{\vee} (a \dot{\vee} b) = \neg a \dot{\vee} (a \dot{\vee} c)$$

Now you try writing a complete proof.

[RR]

[Page 167]

If you were able to complete the proof, have your teacher check your work and then read the material below. If you could not complete the proof, read the material given below.

* * *

Here are the first few steps we wrote as the start of a derivation.

- | | | | |
|-----|---|--------------|---|
| (1) | $a \oplus b = a \oplus c$ | Assumption | |
| | | | $\forall_x \forall_y \forall_z \text{ if } x = y \text{ then } z + x = z + y$ |
| (2) | if $a \oplus b = a \oplus c$ then $-a \oplus (a \oplus b) = -a \oplus (a \oplus c)$ | UI | |
| (3) | $-a \oplus (a \oplus b) = -a \oplus (a \oplus c)$ | (1), (2); MP | |
| | | | $\forall_x \forall_y \forall_z (x \oplus y) \oplus z = x \oplus (y \oplus z)$ |
| (4) | $(-a \oplus a) \oplus b = -a \oplus (a \oplus b)$ | UI | |

At this point we realize that we could use the AXIOMS ' $\forall_x x \oplus -x = 0$ ' and ' $\forall_x x \oplus 0 = x$ '. But we have ' $-a \oplus a$ ' instead of ' $a \oplus -a$ '. We could use the AXIOM ' $\forall_x \forall_y x \oplus y = y \oplus x$ ' to "turn things around" but, in looking ahead we see that we will get ' $0 \oplus b$ ' and then have to use ' $\forall_x \forall_y x \oplus y = y \oplus x$ ' again. Maybe we can write a shorter derivation by first proving that ' $\forall_x \forall_y -x \oplus (x \oplus y) = y$ ' is a theorem.

Turn to PAGE 168.

[RR]

[Page 168]

Here is the start of our proof that ' $\forall_x \forall_y \neg x \dot{\cup} (x \dot{\cup} y) = y$ ' is a theorem.

- | | | | |
|-----|---|---|---------------|
| (1) | $\neg a \dot{\cup} (a \dot{\cup} b) = (a \dot{\cup} b) \dot{\cup} \neg a$ | $\forall_x \forall_y x \dot{\cup} y = y \dot{\cup} x$ | UI |
| (2) | $a \dot{\cup} b = b \dot{\cup} a$ | $\forall_x \forall_y x \dot{\cup} y = y \dot{\cup} x$ | UI |
| (3) | $\neg a \dot{\cup} (a \dot{\cup} b) = (b \dot{\cup} a) \dot{\cup} \neg a$ | | (2), (1); RRE |
| | $\forall_x \forall_y \forall_z (x \dot{\cup} y) \dot{\cup} z = x \dot{\cup} (y \dot{\cup} z)$ | | |
| (4) | $(b \dot{\cup} a) \dot{\cup} \neg a = b \dot{\cup} (a \dot{\cup} \neg a)$ | | UI |
| (5) | $\neg a \dot{\cup} (a \dot{\cup} b) = b \dot{\cup} (a \dot{\cup} \neg a)$ | | (4), (3); RRE |
| (6) | $a \dot{\cup} \neg a = 0$ | $\forall_x x \dot{\cup} \neg x = 0$ | UI |

On your work sheet, you finish the proof.

[RR]

[Page 169]

Completion of proof.

- (7) $\neg a \oplus (a \oplus b) = b \oplus 0$ (6), (5); RRE
-
- (8) $b \oplus 0 = b$ $\forall_x x \oplus 0 = x$ UI
-
- (9) $\neg a \oplus (a \oplus b) = b$ (8), (7); RRE
-
- (10) $\forall_x \forall_y \neg x \oplus (x \oplus y) = y$ (1) - (9); TPP

* * *

Now that we have established that the generalization:

$$(1) \quad \forall_x \forall_y \neg x \oplus (x \oplus y) = y$$

is a theorem, we can use (1) as a premiss in a proof that

' $\forall_x \forall_y \forall_z$ if $x \oplus y = x \oplus z$ then $y = z$ ' is a theorem. Here is a start:

- (1) $a \oplus b = a \oplus c$ Assumption
- $\forall_x \forall_y \forall_z$ if $x = y$ then $z \oplus x = z \oplus y$
- (2) if $a \oplus b = a \oplus c$ then $\neg a \oplus (a \oplus b) = \neg a \oplus (a \oplus c)$ UI
- (3) $\neg a \oplus (a \oplus b) = \neg a \oplus (a \oplus c)$ (1), (2); MP
- (4) $\neg a \oplus (a \oplus b) = b$ $\forall_x \forall_y \neg x \oplus (x \oplus y) = y$ UI
- (5) $b = \neg a \oplus (a \oplus c)$ (4), (3); RRE

On your work sheet, you finish the proof.

[Page 170a]

[RR]

- (1) $a \leftrightarrow b = a \leftrightarrow c$ Assumption *
-
- (2) $\forall x \forall y \forall z$ if $x = y$ then $z + y = z + y$
 if $a \leftrightarrow b = a \leftrightarrow c$ then $\neg a \leftrightarrow (a \leftrightarrow b) = \neg a \leftrightarrow (a \leftrightarrow c)$ UI
-
- (3) $\neg a \leftrightarrow (a \leftrightarrow b) = \neg a \leftrightarrow (a \leftrightarrow c)$ (1), (2); MP
-
- (4) $\forall x \forall y$ $\neg x \leftrightarrow (x \leftrightarrow y) = y$
 $\neg a \leftrightarrow (a \leftrightarrow b) = b$ UI
-
- (5) $b = \neg a \leftrightarrow (a \leftrightarrow c)$ (3), (4); RRE
-
- (6) $\forall x \forall y$ $\neg x \leftrightarrow (x \leftrightarrow y) = y$
 $\neg a \leftrightarrow (a \leftrightarrow c) = c$ UI
-
- (7) $b = c$ (6), (5); RRE
-
- (8) if $a \leftrightarrow b = a \leftrightarrow c$ then $b = c$ C (7); D (1) *
-
- (9) $\forall x \forall y \forall z$ if $x \leftrightarrow y = x \leftrightarrow z$ then $y = z$ (1) - (8); TPP

Note the use of an asterisk to show that the assumption ' $a \leftrightarrow b = a \leftrightarrow c$ ' has been discharged.

* * *

Recall our definition of theorem.

A theorem is a sentence which is the conclusion of a valid derivation whose only premisses are generalizations which we have accepted.

This definition needs some clarification. Now that we have discussed a way to prove conditional sentences by taking the antecedent as an assumption and then discharging that assumption, we need to amend the definition to:

A theorem is a sentence which is the conclusion of a valid derivation whose only undischarged premisses are sentences which we have accepted.

Turn to PAGE 170.

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[RR]

[Page 170]

The phrase 'sentences we have accepted' refers to our principles, the sentence ' $\forall_x x = x$ ' and theorems previously established.

When considering a way to prove that a given sentence is a theorem, one often sees that another sentence [$\forall_x \forall_y -x \neq (x \neq y) = y$ in our last proof] would be very useful. If a sentence which you wish to use as a premiss has not been established as a theorem it is proved before deriving the given sentence. The second sentence may then be used as a premiss in a proof that the given sentence is a theorem.

[RR]

[Page 171]

Pete and the others returned to Zabbranchburg. His friends in Zabbranchburg High were interested in hearing about Pete's experiences on Glox. He told them what he had learned about the Gloxian Axioms and about the theorems he and the others had derived.

Some of the mathematics students at Zabbranchburg wondered if they could prove that other sentences were Gloxian theorems. Since they were not very familiar with the Gloxian symbols ' \forall ' and ' \cdot ', they decided to use their basic principles and then translate to Gloxian symbols.

Bruce noticed the theorem:

$$\forall_x \forall_y \forall_z \text{ if } x = y \text{ then } z + x = z + y$$

and decided that he could prove that the sentence:

$$\forall_x \forall_y \forall_z \text{ if } x = y \text{ then } x + z = y + z$$

is a theorem. Can you derive the sentence ' $\forall_x \forall_y \forall_z \text{ if } x = y \text{ then } x + z = y + z$ '? Try it on your work sheet. You may write either a column derivation or a tree-chart derivation. You may need to write an informal plan first. [Hint: Notice that the pattern sentence is a conditional. Consider using the antecedent [$a = b$] of the pattern sentence as an assumption in your derivation.]

[RR]

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I Column derivation

(1)	$a = b$	Assumption *
(2)	$a + c = a + c$	$\forall_x x = x$ UI
(3)	$a + c = b + c$	(1), (2); RRE
(4)	if $a = b$ then $a + c = b + c$	C(3); D(1) *
(5)	$\forall_x \forall_y \forall_z$ if $x = y$ then $x + z = y + z$	(1) - (4); TPP

II Tree-diagram

*	$\forall_x x = x$	UI
$a = b$	$a + c = a + c$	RRE
	$a + c = b + c$	C; D*
	if $a = b$ then $a + c = b + c$	TPP
$\forall_x \forall_y \forall_z$	if $x = y$ then $x + z = y + z$	

Your derivation may be different from ours. Consult with your teacher if you have any doubts.

* * *

Prove that the sentence

$$\forall_x \forall_y \forall_z \text{ if } x = y \text{ then } xz = yz$$

is a theorem.

Turn to PAGE 173a.

[Page 173a]

[RR]

Have your teacher check your work.

* * *

Here are several more sentences which Bruce was able to derive from the basic principles and whose corresponding Gloxian sentences are Gloxian theorems.

$$(1) \quad \forall_x \forall_y \text{ if } x = y \text{ then } -x = -y$$

$$(2) \quad \forall_x \forall_y \text{ if } x + y = 0 \text{ then } x = -y$$

$$(3) \quad \forall_x \forall_y \text{ if } x + y = 0 \text{ then } -x = y$$

Soon after deriving theorems (1), (2), and (3), Bruce had a chance to use one of them. Ann and Lois were arguing about the Gloxian numeral:

$$-(2 \oplus 3)$$

Ann claimed that

$$-(2 \oplus 3) = -2 \oplus 3$$

and Lois claimed that

$$-(2 \oplus 3) = -2 \oplus -3$$

SKIP THIS PAGE

[RR]

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Read the facing page first.

Bruce said, "Look I'll settle it for you. You accept the Gloxian AXIOMS and the theorems we have derived. So, you must accept any consequences of those AXIOMS and theorems. Consider the Gloxian equivalent of theorem (3). An instance of theorem (3) is:

$$(3') \text{ if } (2 \oplus 3) \oplus (-2 \oplus -3) = 0 \text{ then } -(2 \oplus 3) = -2 \oplus -3$$

Now, I can derive the antecedent of (3') from Gloxian AXIOMS and theorems. Then by modus ponens, Lois's sentence follows. So, Lois is right."

Answer on your work sheet.

- (a) Which of the theorems (1), (2), and (3), would you use to derive the sentence ' $(-a)b = -(ab)$ '?
- (b) What instance would you use?
- (c) Which of the theorems would you use to derive the sentence ' $--a = a$ '?
- (d) What instance would you use?

[RR]

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- (a) $\forall x \forall y$ if $x + y = 0$ then $x = -y$
 (b) if $(-a)b + ab = 0$ then $(-a)b = -(ab)$
 (c) $\forall x \forall y$ if $x + y = 0$ then $-x = y$
 (d) if $-a + a = 0$ then $--a = a$

* * *

In order to derive the sentence ' $(-a) \cdot b = -(a \cdot b)$ ', we would first derive the sentence ' $(-a) \cdot b + a \cdot b = 0$ ' and then use 'if $(-a) \cdot b + a \cdot b = 0$ then $(-a) \cdot b = -(a \cdot b)$ '. Here is a derivation of the theorem ' $\forall x \forall y (-x) \cdot y = -(xy)$ '.

- | | | | |
|------|---|-------------------------------------|-----------------|
| (1) | $(-a) \cdot b + a \cdot b = (-a) \cdot b + a \cdot b$ | $\forall x x = x$ | UI |
| | <hr/> | | |
| | $\forall x \forall y \forall z (x + y) \cdot z = x \cdot z + y \cdot z$ | | |
| (2) | $(-a + a) \cdot b = (-a) \cdot b + a \cdot b$ | | UI |
| | <hr/> | | |
| (3) | $(-a) \cdot b + a \cdot b = (-a + a) \cdot b$ | | (2), (1); RRE |
| | <hr/> | | |
| (4) | $-a + a = a + -a$ | $\forall x \forall y x + y = y + x$ | UI |
| | <hr/> | | |
| (5) | $(-a) \cdot b + a \cdot b = (a + -a) \cdot b$ | | (4), (3); RRE |
| | <hr/> | | |
| (6) | $a + -a = 0$ | $\forall x x + -x = 0$ | UI |
| | <hr/> | | |
| (7) | $(-a) \cdot b + a \cdot b = 0 \cdot b$ | | (6), (5); RRE |
| | <hr/> | | |
| (8) | $0 \cdot b = 0$ | $\forall x 0 \cdot x = 0$ | UI |
| | <hr/> | | |
| (9) | $(-a) \cdot b + a \cdot b = 0$ | | (8), (7); RRE |
| | <hr/> | | |
| | $\forall x \forall y$ if $x + y = 0$ then $x = -y$ | | |
| (10) | if $(-a) \cdot b + a \cdot b = 0$ then $(-a) \cdot b = -(a \cdot b)$ | | UI |
| | <hr/> | | |
| (11) | $(-a) \cdot b = -(a \cdot b)$ | | (9), (10); MP |
| | <hr/> | | |
| (12) | $\forall x \forall y (-x) \cdot y = -(x \cdot y)$ | | (1) - (11); TPP |

On your work sheet, prove that ' $\forall x --x = x$ ' is a theorem.

[RR]

[Page 175]

Here is our proof:

- | | | | |
|-----|--------------------------------|---|----------------|
| (1) | $-a + a = a + -a$ | $\forall_x \forall_y \ x + y = y + x$ | UI |
| (2) | $a + -a = 0$ | $\forall_x \ x + -x = 0$ | UI |
| (3) | $-a + a = 0$ | | (2), (1); RRE |
| (4) | if $-a + a = 0$ then $--a = a$ | $\forall_x \forall_y \ \text{if } x + y = 0 \text{ then } -x = y$ | UI |
| (5) | $--a = a$ | | (3), (4); MP |
| (6) | $\forall_x \ --x = x$ | | (1) - (5); TPP |

Your derivation may be different from ours. If it is, have your teacher check it.

* * *

We now have another way to establish a pattern sentence. For example, if you wish to establish the pattern sentence ' $--a = a$ ', look for a theorem which has as an instance a conditional 'if $-a + a = 0$ then $--a = a$ ' whose consequent ' $--a = a$ ' is the sentence you want. Then, if you can derive the antecedent ' $-a + a = 0$ ', you can use modus ponens.

Prove that the sentence:

$$\forall_x \ -x = (-1) \cdot x$$

is a theorem. Consider the conditional sentence:

$$\text{if } a + (-1) \cdot a = 0 \text{ then } -a = (-1) \cdot a$$

[RR]

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Have your teacher check your proof.

* * *

Many of the sentences we use are simple sentences like:

(1) 5 is a prime number (2) $2 > 5$ (3) John is absent

and other sentences are complex like:

(3) if $5 > 2$ then $5 > 1$

(4) $5 > 2$ or $5 > 1$

(5) $5 > 2$ and $5 > 1$

(6) not $(2 > 5)$

Note that we have used simple sentences together with phrases:

if then --- ; or ; and ; not

to form complex sentences. We could also form complex sentences whose components are also complex sentences. For example:

(7) if $(5 > 2 \text{ and } 2 > 1)$ then $5 > 1$

(8) not $(5 > 2 \text{ or } 5 > 1)$

Each of the sentences (3), (4), and (5) has two components, the sentences:

$5 > 2$

$5 > 1$

and sentence (6) has one component:

$2 > 5$

The components of a complex sentence may themselves be simple sentences [as in (3) - (6)] or complex sentences [as in (7) and (8)].

Complete on your work sheet:

The first component of a conditional sentence is called the _____?
and the second component is called the _____?.

[RR]

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The first component of a conditional sentence is called the antecedent and the second component is called the consequent.

* * *

On your work sheet, write 'True' or 'False'.

- (a) 5 is a prime number
- (b) 5 is not a prime number
- (c) $-3 = -3$
- (d) $-3 \neq -3$
- (e) $2 > 5$
- (f) $2 \not> 5$

[RR]

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- True (a) 5 is a prime number
False (b) 5 is not a prime number
True (c) $-3 = -3$
False (d) $-3 \neq -3$
False (e) $2 > 5$
True (f) $2 \not> 5$

* * *

In our formal language we write the denial of ' $-3 = -3$ ' as ' $\text{not } (-3 = -3)$ '.
Hence, in our formal language, the sentence:

$\text{not}(5 \text{ is a prime number})$

is the denial of:

5 is a prime number.

Here are three sentences. Write, on your work sheet, the denial of each sentence. Use the formal language.

- (a) $5 + 7 = 7 + 5$
(b) $5 \cdot 0 = 5$
(c) $\text{not}(2 + 3 = 5)$

[RR]

[Page 179]

- (a) The denial of ' $5 + 7 = 7 + 5$ ' is: $\text{not}(5 + 7 = 7 + 5)$
- (b) The denial of ' $5 \cdot 0 = 5$ ' is: $\text{not}(5 \cdot 0 = 5)$
- (c) The denial of ' $\text{not}(2 + 3 = 5)$ ' is: $\text{not}[\text{not}(2 + 3 = 5)]$

* * *

You may have written ' $2 + 3 = 5$ ' as your answer for (c). The sentence ' $2 + 3 = 5$ ' is not a denial sentence. A denial sentence is a sentence which begins with not.

Sentences like:

$\text{not}(5 + 7 = 7 + 5)$ $\text{not}(5 \cdot 0 = 5)$

do not often appear in ordinary English. Instead of writing ' $\text{not}(5 + 7 = 7 + 5)$ ' as the contradictory of ' $5 + 7 = 7 + 5$ ', most people would write:

$5 + 7 \neq 7 + 5$

Similarly, the sentence:

(1) John is not absent

is the contradictory of:

(2) John is absent

Two sentences are contradictories whenever it is the case that if one is true the other must be false and at least one of the sentences is true.

(3) This house is black

(4) This house is not black

are contradictories.

Turn to PAGE 180.

[RR]

[Page 180]

The denial of:

$$\text{not}(2 + 3 = 5)$$

is:

$$\text{not}[\text{not}(2 + 3 = 5)].$$

The contradictory of:

$$(1) \quad 2 + 3 = 5$$

is:

$$(2) \quad 2 + 3 \neq 5$$

and (1) is the contradictory of (2).

We believe that most people consider the sentence ' $2 + 3 = 5$ ' to be a logical consequence of the sentence ' $\text{not}[\text{not}(2 + 3 = 5)]$ '. That is, the inference:

$$(1) \quad \frac{\text{not}[\text{not}(2 + 3 = 5)]}{2 + 3 = 5}$$

is considered to be a valid inference. Also, we believe that most people consider that the inference:

$$(2) \quad \frac{2 + 3 = 5}{\text{not}[\text{not}(2 + 3 = 5)]}$$

is valid.

On your work sheet, complete each inference.

$$(a) \quad \frac{\text{not}[\text{not}(5 + 7 = 12)]}{?}$$

$$(b) \quad \frac{4 + 7 = 11}{?}$$

$$(c) \quad \frac{\text{not}[\text{not}(5 > 3)]}{?}$$

$$(d) \quad \frac{5 > 3}{?}$$

[RR]

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$$(a) \frac{\text{not} [\text{not} (5 + 7 = 12)]}{5 + 7 = 12}$$

$$(b) \frac{4 + 7 = 11}{\text{not} [\text{not} (4 + 7 = 11)]}$$

$$(c) \frac{\text{not} [\text{not} (5 > 3)]}{5 > 3}$$

$$(d) \frac{5 > 3}{\text{not} [\text{not} (5 > 3)]}$$

* * *

The rule of inference which justifies inferences (b) and (d) is called double denial.

Double Denial (DD)

$$\frac{p}{\text{not} [\text{not} (p)]}$$

is a valid inference pattern

The rule of inference which justifies inferences (a) and (c) is called reverse double denial.

Reverse Double Denial (RDD)

$$\frac{\text{not} [\text{not} (p)]}{p}$$

is a valid inference pattern

Use the rule of Reverse Double Denial and complete each of these inferences.

$$(a) \frac{\text{not} [\text{not} (5 > 2)]}{?}$$

$$(b) \frac{\text{not} (7 \neq 4 + 3)}{?}$$

[RR]

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$$(a) \frac{\text{not} [\text{not} (5 > 2)]}{5 > 2}$$

$$(b) \frac{\text{not} (7 \neq 4 + 3)}{7 = 4 + 3}$$

Technically, the premiss in (b) should be 'not[not(7 = 4 + 3)]' in order to use RDD. However, we will use both DD and RDD with sentences like '7 ≠ 4 + 3'.

* * *

The open sentences:

$$(1) \quad a + b = 0$$

$$(2) \quad a + b \neq 0$$

are neither true nor false. However, if we substitute numerals for 'a' and 'b' in sentences (1) and (2), we get a pair of sentences which are contradictories. Hence, we will also refer to pattern sentences like (1) and (2) as contradictories.

For each of the sentences given below, write its contradictory on your work sheet.

(a) John is sick

(b) $a + c \neq d$

[RR]

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(a) John is not sick

(b) $a + c = d$

* * *

Decide which of these inferences are correct. Indicate your choice by writing the appropriate letters on your work sheet.

- (a) John is sick if John is sick then John is absent
John is absent
- (b) not(John is sick) if John is sick then John is absent
John is absent
- (c) not(John is sick) if John is sick then John is absent
not(John is absent)
- (d) if John is sick then John is absent John is absent
John is sick
- (e) if John is sick then John is absent not(John is absent)
not(John is sick)

[RR]

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- (a) $\frac{\text{John is sick} \quad \text{if John is sick then John is absent}}{\text{John is absent}}$
- (b) $\frac{\text{not (John is sick)} \quad \text{if John is sick then John is absent}}{\text{John is absent}}$ **INVALID**
- (c) $\frac{\text{not (John is sick)} \quad \text{if John is sick then John is absent}}{\text{not (John is absent)}}$ **INVALID**
- (d) $\frac{\text{if John is sick then John is absent} \quad \text{John is absent}}{\text{John is sick}}$ **INVALID**
- (e) $\frac{\text{if John is sick then John is absent} \quad \text{not (John is absent)}}{\text{not (John is sick)}}$

* * *

Decide which of the inferences are correct. Indicate your choice by writing the appropriate letters on your work sheet.

- (a) $\frac{\text{if } a = b \text{ then } ac = bc \quad ac \neq bc}{a \neq b}$
- (b) $\frac{\text{if } a = b \text{ then } -a = -b \quad -a \neq -b}{a \neq b}$
- (c) $\frac{\text{if } A \in \overline{BC} \text{ then } B \notin \overline{AC} \quad B \in \overline{AC}}{A \notin \overline{BC}}$
- (d) $\frac{\text{if } A \in \overline{BC} \text{ then } A \in \overline{BC} \quad A \notin \overline{BC}}{A \notin \overline{BC}}$
- (e) $\frac{\text{if } A \in \overline{BC} \text{ then } A \in \overline{BC} \quad A \in \overline{BC}}{A \in \overline{BC}}$

Turn to PAGE 185a.

[Page 185a]

[RR]

- (a) $\frac{\text{if } a = b \text{ then } ac = bc}{a \neq b} \quad ac \neq bc$
- (b) $\frac{\text{if } a = b \text{ then } -a = -b}{a \neq b} \quad -a \neq -b$
- (c) $\frac{\text{if } A \in \overline{BC} \text{ then } B \notin \overline{AC}}{A \notin \overline{BC}} \quad B \in \overline{AC}$
- (d) $\frac{\text{if } A \in \overline{BC} \text{ then } A \notin \overline{BC}}{A \notin \overline{BC}} \quad A \notin \overline{BC}$
- (e) $\frac{\text{if } A \in \overline{BC} \text{ then } A \in \overline{BC}}{A \in \overline{BC}} \quad A \in \overline{BC}$

Turn to PAGE 96.

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[RR]

[Page 185]

Inferences (a), (b), and (c) are justified by the rule of reasoning called modus tollendo tollens, or for short, modus tollens. Modus tollens tells you that you can infer the denial of the antecedent of a conditional from the conditional together with the denial of its consequent.

Schematically,

<p>MODUS TOLLENS (MT)</p> $\frac{\text{if } p \text{ then } q \quad \text{not } q}{\text{not } p}$ <p>is a valid inference pattern.</p>

Since we will more often write a sentence like:

$$a \neq 0$$

instead of:

$$\text{not}(a = 0)$$

we shall understand that modus tollens tells us that we can infer the contradictory of the antecedent of a conditional from the conditional together with the contradictory of its consequent.

Decide which of these inferences are valid, and then write 'MP' or 'MT' to indicate the rule of reasoning which justifies the inference.

(a) $\frac{\text{if } a = 0 \text{ then } ab = 0 \quad ab \neq 0}{a \neq 0}$

(b) $\frac{\text{if } a \neq 0 \text{ then } ab \neq 0 \quad ab = 0}{a = 0}$

(c) $\frac{\text{if } a \neq 0 \text{ then } ab = 0 \quad a \neq 0}{ab = 0}$

[RR]

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(a) MT

(b) MT

(c) MP

* * *

We can use several of our rules of reasoning in a single derivation. Here is a sample:

$$\begin{array}{l}
 \frac{a = 2 \quad a^2 - 3a - 6 = 0}{2^2 - 3 \cdot 2 - 6 = 0} \text{RRE} \\
 \frac{\text{if } a = 2 \text{ then } 2^2 - 3 \cdot 2 - 6 = 0 \quad C, D^*}{a \neq 2 \quad 2^2 - 3 \cdot 2 - 6 \neq 0} \text{MT}
 \end{array}$$

Each of the following exercises gives a pattern for a sequence of inferences. On your work sheet, write sentences which complete the pattern and indicate the rule of reasoning which justifies each inference.

$$\begin{array}{l}
 \frac{\text{if } a = 0 \text{ then } ab = 0 \quad ab \neq 0}{?} \quad \text{if } a \neq 0 \text{ then } a^2 > 0 \\
 \frac{?}{?} *
 \end{array}$$

$$\begin{array}{l}
 \frac{\text{if } A \in \overline{BC} \text{ then } A \in \overline{BC} \quad A \notin \overline{BC}}{?} * \\
 ?
 \end{array}$$

$$\begin{array}{l}
 \frac{\text{if } p \text{ then } q \quad \text{not } q}{\text{not } p} * \\
 ?
 \end{array}$$

[RR]

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$$\begin{array}{l}
 \text{(a) } \frac{\frac{\text{if } a = 0 \text{ then } ab = 0 \quad ab \neq 0^*}{a \neq 0} \text{ MT} \quad \text{if } a \neq 0 \text{ then } a^2 > 0}{a^2 > 0} \text{ MP} \\
 \hline
 \text{if } ab \neq 0 \text{ then } a^2 > 0 \quad \text{C, D}^*
 \end{array}$$

$$\begin{array}{l}
 \text{(b) } \frac{\frac{\text{if } A \in \overline{BC} \text{ then } A \in \overline{BC} \quad A \notin \overline{BC}^*}{A \notin \overline{BC}} \text{ MT} \quad \text{if } A \notin \overline{BC} \text{ then } A \notin \overline{BC}}{A \notin \overline{BC}} \text{ C, D}^*
 \end{array}$$

$$\begin{array}{l}
 \text{(c) } \frac{\frac{\text{if } p \text{ then } q \quad \text{not } q^*}{\text{not } p}}{\text{if not } q \text{ then not } p} \text{ C, D}^*
 \end{array}$$

* * *

Consider the sentence:

(1) If John is sick then John is absent

On your work sheet, write the letters which correspond with sentences which you think are consequences of (1).

(a) If John is not sick then John is not absent.

(b) If John is absent then John is sick.

(c) If John is not absent then John is not sick.

[RR]

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Only

(c) If John is not absent then John is not sick.

is a consequence of:

(1) If John is sick then John is absent.

* * *

We have said that the sentence:

(c) if John is not absent then John is not sick

is a consequence of:

(1) if John is sick then John is absent

Let's try to write a derivation which has (c) as conclusion and (1) as premiss. Since the sentence we want to derive (c) is a conditional sentence, it may help to take its antecedent ['John is not absent'] as a premiss which will be discharged.

Here is the beginning of a derivation of sentence (1).

if John is sick then John is absent	John is not absent [*]	
?		MT
?		C, D*

On your work sheet, complete the derivation.

[RR]

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if John is sick then John is absent	*
John is not sick	John is not absent
if John is not absent then John is not sick	MT C; D *

* * *

The derivation above shows that the conditional sentence:

(1) if John is not absent then John is not sick

is a logical consequence of the conditional sentence:

(2) if John is sick then John is absent

The conditional sentence (1) is called the contrapositive of (2).

For each of these conditional sentences, write its contrapositive.

(a) if $a = 2$ then $a^2 = 4$

(b) if $a > 0$ then $a^2 > 0$

(c) if $a = 0$ then $ab = 0$

[RR]

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(a) if $a^2 \neq 4$ then $a \neq 2$ (b) if $a^2 \neq 0$ then $a \neq 0$ (c) if $ab \neq 0$ then $a \neq 0$

* * *

Here is the start of a derivation of the sentence:

if $a^2 \neq 4$ then $a \neq 2$

(1)	if $a = 2$ then $a^2 = 4$	<u>Premiss</u>
(2)	$a^2 \neq 4$	<u>Assumption*</u>
(3)	$a^2 \neq 2$	<u>(1), (2); MT</u>
(4)	?	<u>C (3), D (2)*</u>

On your work sheet, write the last line of the derivation.

[RR]

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- | | | |
|-----|---------------------------------|---------------|
| (1) | if $a = 2$ then $a^2 = 4$ | Premiss |
| (2) | $a^2 \neq 4$ | Assumption* |
| (3) | $a \neq 2$ | (1), (2); MT |
| (4) | if $a^2 \neq 4$ then $a \neq 2$ | C (3), D (2)* |

* * *

The derivation above shows that the sentence:

- (1) if $a^2 \neq 4$ then $a \neq 2$

is a logical consequence of:

- (2) if $a = 2$ then $a^2 = 4$

Is it the case that each conditional sentence implies its contrapositive? The answer is 'Yes'. Each inference of this kind can be justified in the same way we justified the inference of (1) from (2):

$$\begin{array}{rcl}
 \text{if } p \text{ then } q & \text{not } q & * \\
 \hline
 & \text{not } p & \text{MT} \\
 \hline
 \text{if not } q \text{ then not } p & & \text{C, D*}
 \end{array}$$

So, we have justified a new rule of reasoning -- the rule of contraposition.

Contraposition

$$\begin{array}{rcl}
 \text{if } p \text{ then } q & & \\
 \hline
 \text{if not } q \text{ then not } p & &
 \end{array}$$

is a valid inference pattern.

Turn to PAGE 192.

[RR]

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Let's make use of the rule of contraposition:

$$\frac{\text{if } p \text{ then } q}{\text{if not } q \text{ then not } p}$$

in proving the theorem:

$$\forall x \forall y \text{ if } xy \neq 0 \text{ then } y \neq 0$$

The next-to-last step in such a proof will be a conditional sentence, say:

$$\text{if } ab \neq 0 \text{ then } b \neq 0$$

The rule of contraposition suggests that we try to derive the conditional sentence:

$$\text{if } b = 0 \text{ then } ab = 0,$$

and then use contraposition. So, now our job is to derive 'ab = 0' from the assumption 'b = 0'.

Here is an incomplete derivation:

(1)	b = 0	Assumption *
(2)	a • 0 = 0	$\forall_x x \cdot 0 = 0$ UI
(3)	?	(1), (2); RRE
(4)	if b = 0 then ab = 0	C (3), D (1)*
(5)	?	(4); Contraposition
(6)	?	(1) - (5); TPP

On your work sheet, write the missing sentences.

[RR]

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- | | | |
|-----|---|------------------------------|
| (1) | $b = 0$ | Assumption * |
| (2) | $a \cdot 0 = 0$ | $\forall_x x \cdot 0 = 0$ UI |
| (3) | $a \cdot b = 0$ | (1), (2); RRE |
| (4) | if $b = 0$ then $a \cdot b = 0$ | C (3); D (1) * |
| (5) | if $a \cdot b \neq 0$ then $b \neq 0$ | (4); Contraposition |
| (6) | $\forall_x \forall_y$ if $x \cdot y \neq 0$ then $y \neq 0$ | (1) - (5); TPP |

* * *

Here is a derivation of the sentence:

if $a + 3 = 1$ then $a \neq 2$

	$a + 3 = 1$	DD
	$\text{not}(a + 3 \neq 1)$	MT
if $a = 2$ then $a + 3 \neq 1$	$a \neq 2$	
	if $a + 3 = 1$ then $a \neq 2$	C, D *

On your work sheet, complete this inference pattern:

	q	DD
	$\text{not}(\text{not } q)$	MT
if p then not q	?	
	?	C, D *

[RR]

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	*	
	q	
if p then not q	not (not q)	DD
	not p	MT
		C, D*
	if q then not p	

* * *

The pattern given above shows that a sentence of the form:

if q then not p

is a consequence of a sentence of the form:

if p then not q

We shall call this a symmetric rule of contraposition:

if p then not q
 if q then not p

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[RR]

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Write, on your work sheet sentences which complete the derivations and tell what rule of reasoning justifies each step.

$$\begin{array}{lcl}
 \text{(a)} & \text{if } a = 2 \text{ then } a + 3 \neq 1 & \frac{a + 3 = 1}{\text{not}(a + 3 \neq 1)} \quad ? \\
 & \underline{\hspace{10em}} & \underline{\hspace{10em}} \quad ? \\
 & \hspace{10em} ? & \hspace{10em} ? *
 \end{array}$$

$$\begin{array}{lcl}
 \text{(b)} & \text{if } a \neq 2 \text{ then } a + 3 = 1 & a + 3 \neq 1 \quad ? \\
 & \underline{\hspace{10em}} \quad \text{not}(a \neq 2) & \underline{\hspace{10em}} \quad ? \\
 & \underline{\hspace{10em}} \quad ? & \underline{\hspace{10em}} \quad ? * \\
 & \hspace{10em} ? &
 \end{array}$$

$$\begin{array}{lcl}
 \text{(c)} & \text{if } a \neq 2 \text{ then } a + 3 \neq 1 & \frac{a + 3 = 1}{\text{not}(a + 3 \neq 1)} \quad ? \\
 & \underline{\hspace{10em}} \quad \text{not}(a \neq 2) & \underline{\hspace{10em}} \quad ? \\
 & \underline{\hspace{10em}} \quad ? & \underline{\hspace{10em}} \quad ? \\
 & \hspace{10em} ? & \hspace{10em} ? *
 \end{array}$$

$$\begin{array}{lcl}
 \text{(d)} & \text{if } a = 2 \text{ then } a + 3 = 1 & a + 3 \neq 1 \quad ? \\
 & \underline{\hspace{10em}} \quad ? & \underline{\hspace{10em}} \quad ? * \\
 & \hspace{10em} ? &
 \end{array}$$

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[RR]

$$\begin{array}{lcl}
 \text{(a)} & \frac{\text{if } a = 2 \text{ then } a + 3 \neq 1 \quad \frac{a + 3 = 1}{\text{not}(a + 3 \neq 1)}}{a \neq 2} & \begin{array}{l} \text{DD} \\ \text{MT} \\ \text{C, D*} \end{array} \\
 & \hline
 & \text{if } a + 3 = 1 \text{ then } a \neq 2
 \end{array}$$

$$\begin{array}{lcl}
 \text{(b)} & \frac{\text{if } a \neq 2 \text{ then } a + 3 = 1 \quad \frac{a + 3 \neq 1}{\text{not}(a \neq 2)}}{a = 2} & \begin{array}{l} \text{MT} \\ \text{RDD} \\ \text{C, D*} \end{array} \\
 & \hline
 & \text{if } a + 3 \neq 1 \text{ then } a = 2
 \end{array}$$

$$\begin{array}{lcl}
 \text{(c)} & \frac{\text{if } a \neq 2 \text{ then } a + 3 \neq 1 \quad \frac{a + 3 = 1}{\text{not}(a + 3 \neq 1)}}{\text{not}(a \neq 2)} & \begin{array}{l} \text{DD} \\ \text{MT} \\ \text{RDD} \\ \text{C, D*} \end{array} \\
 & \hline
 & a = 2 \\
 & \hline
 & \text{if } a + 3 = 1 \text{ then } a = 2
 \end{array}$$

$$\begin{array}{lcl}
 \text{(d)} & \frac{\text{if } a = 2 \text{ then } a + 3 = 1 \quad \frac{a + 3 \neq 1}{\text{not}(a + 3 \neq 1)}}{a \neq 2} & \begin{array}{l} \text{MT} \\ \text{C, D*} \end{array} \\
 & \hline
 & \text{if } a + 3 \neq 1 \text{ then } a \neq 2
 \end{array}$$

* * *

Derivations like (a) can be represented by:

$$\begin{array}{lcl}
 \text{(a')} & \frac{\text{if } p \text{ then not } q \quad \frac{q}{\text{not}[\text{not } q]}}{\text{not } p} & * \\
 & \hline
 & \text{if } q \text{ then not } p
 \end{array}$$

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[RR]

[Page 196]

The inference pattern (a') shows that a sentence of the form:

if q then not p

is a logical consequence of a sentence of the form:

if p then not q

Hence, we write:

$$\frac{\text{if p then not q}}{\text{if q then not p}}$$
 is a valid inference pattern.

Thus, the inference:

$$\frac{\text{if today is Tuesday then tomorrow is not Wednesday}}{\text{if tomorrow is Wednesday then today is not Tuesday}}$$

is valid. [What sentence is substituted for 'p'? What sentence is substituted for 'q' ?]

On your work sheet, complete these schematic representatives of the inferences (b), (c), and (d).

(b')
$$\frac{\text{if not p then q} \quad \frac{*}{?}}{\frac{\text{not (not p)} \quad ?}{?} \quad ?} \quad ? *$$

(c')
$$\frac{\frac{? \quad \frac{\frac{*}{q}}{\text{not [not q]}} \quad ?}{\text{not [not p]} \quad ?} \quad ?}{\text{if q then p}} \quad ? *$$

(d')
$$\frac{\text{if p then q} \quad \frac{*}{?}}{\frac{?}{?} \quad ?} \quad ? *$$

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$$\begin{array}{l}
 \text{(b')} \quad \frac{\text{if not } p \text{ then } q \quad \frac{\text{not } q}{*} \text{ MT}}{\frac{\text{not (not } p)}{\text{p}} \text{ RDD}} \text{ C, D } *
 \end{array}$$

$$\begin{array}{l}
 \text{(c')} \quad \frac{\text{if not } p \text{ then not } q \quad \frac{\frac{q}{*} \text{ DD}}{\text{not [not } q]} \text{ MT}}{\frac{\text{not [not } p]}{\text{p}} \text{ RDD}} \text{ C, D } *
 \end{array}$$

$$\begin{array}{l}
 \text{(d')} \quad \frac{\text{if } p \text{ then } q \quad \frac{\text{not } q}{*} \text{ MT}}{\text{not } p} \text{ C, D } * \\
 \text{if not } q \text{ then not } p
 \end{array}$$

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By (a'), (b'), (c'), and (d') we have justified the four rules of contraposition given by the inference patterns:

$$(a'') \frac{\text{if } p \text{ then not } q}{\text{if } q \text{ then not } p}$$

$$(b'') \frac{\text{if not } p \text{ then } q}{\text{if not } q \text{ then } p}$$

$$(c'') \frac{\text{if not } p \text{ then not } q}{\text{if } q \text{ then } p}$$

$$(d'') \frac{\text{if } p \text{ then } q}{\text{if not } q \text{ then not } p}$$

Whenever we write an inference which is an example of any of the schemes (a''), (b''), (c''), (d''), we will write 'Cp' (for 'contraposition') as the justification. For example:

$$\frac{\text{if } a = 0 \text{ then } a^2 \neq 0}{\text{if } a^2 > 0 \text{ then } a \neq 0} \text{ Cp}$$

On your work sheet, write a derivation of the sentence:

$$\forall_x \forall_y \text{ if } x + y \neq 0 \text{ then } y \neq -x$$

[Hint: Use 'b = -a' as an assumption; derive the sentence 'a + b = 0'; conditionalize and use contraposition.]

[RR]

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Either of the derivations below is correct.

I

*	$\forall_x x + -x = 0$	
$b = -a$	$a + -a = 0$	UI
$a + b = 0$		RRE
if $b = -a$ then $a + b = 0$		C, D*
if $a + b \neq 0$ then $b \neq -a$		Cp
$\forall_x \forall_y$ if $x + y \neq 0$ then $y \neq -x$		TPP

II

(1)	$a + -a = 0$	$\forall_x x + -x = 0$	UI
(2)	$b = -a$	Assumption *	
(3)	$a + b = 0$		(2), (1); RRE
(4)	if $b = -a$ then $a + b = 0$		C(3), D(2)*
(5)	if $a + b \neq 0$ then $b \neq -a$		(4); Cp
(6)	$\forall_x \forall_y$ if $x + y \neq 0$ then $y \neq -x$		(1) - (5); TPP

* * *

Consider the conditional sentence:

- (1) if John is sick then John is absent

Given below are three conditional sentences. For each sentence which is a logical consequence of (1), write the corresponding letter on your work sheet.

- (a) if John is absent then John is sick
 (b) if John is not sick then John is not absent
 (c) if John is not absent then John is not sick

[RR]

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Only

(c) If John is not absent then John is not sick

is a logical consequence of:

(1) If John is sick then John is absent

* * *

There are two fallacies which are somewhat like contraposition:

~~if p then q~~
~~if q then p~~ **INVALID**
~~if p then q~~
~~if not p then not q~~ **INVALID**

Each is closely related to these two fallacies:

~~if p then q~~
~~p~~ **INVALID**
 [affirming the consequent]

~~not p~~ ~~if p then q~~
~~not q~~ **INVALID**
 [denying the antecedent]

People who commit the fallacy of affirming the consequent probably do so because they think [incorrectly] that a conditional sentence implies its converse. [The converse of a conditional is what you get when you interchange the antecedent and the consequent.] For example, they may think that the conditional sentence:

(1) If John is sick then John is absent

implies its converse:

(2) If John is absent then John is sick.

[Of course, (1) does not imply (2). The first sentence may be true and the second false. John may be absent and not sick. He may be at the movies.]

On your work sheet, write the converse of

- (a) if John is sick then John is absent
- (b) if $a + b = 0$ then $-a = b$
- (c) if $a \neq b$ then $ac \neq bc$

[RR]

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(a') if John is absent then John is sick

(b') if $-a = b$ then $a + b = 0$

(c') if $ac \neq bc$ then $a \neq b$

* * *

Consider these three sentences:

(1) if $-a = b$ then $a + b = 0$

(2) if $a + b = 0$ then $-a = b$

(3) if $a + b \neq 0$ then $-a \neq b$

(2) is the converse of (1); (3) is the contrapositive of (1); and (1) is the converse of (2). Remember, the converse of a conditional is what you get when you interchange the antecedent and the consequent.

On your work sheet, write the converse of each conditional.

(a) if John is absent then John is sick

(b) if today is Tuesday then tomorrow is Wednesday

(c) if $-3 = -3$ then $-3 + 7 = -3 + 7$

[RR]

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- (a) If John is sick then John is absent.
 (b) If tomorrow is Wednesday then today is Tuesday.
 (c) If $-3 + 7 = -3 + 7$ then $-3 = -3$.

* * *

Since any instance of the generalization:

$$(1) \forall_x \forall_y \forall_z \text{ if } x = y \text{ then } x + z = y + z$$

is the converse of the corresponding instance of:

$$(2) \forall_x \forall_y \forall_z \text{ if } x + z = y + z \text{ then } x = y,$$

we will say that (1) is the converse of (2) [and (2) is the converse of (1)].

Similarly, the universal generalization:

$$\forall_x \forall_y \forall_z \text{ if } x \neq y \text{ then } x + z \neq y + z$$

is the contrapositive of:

$$\forall_x \forall_y \forall_z \text{ if } x + z = y + z \text{ then } x = y.$$

Each exercise contains a conditional sentence or a universal generalization of a conditional sentence. Write the contrapositive and the converse.

(a) if $A \in \overrightarrow{BC}$ then $A \in \overleftarrow{BC}$

Contrapositive: ?

Converse: ?

(b) $\forall_x \forall_y \forall_z \text{ if } x = y \text{ then } xz = yz$

Contrapositive: ?

Converse: ?

(c) $\forall_x \forall_y \text{ if } x \neq 0 \text{ then } xy \neq 0$

Contrapositive: ?

Converse: ?

[RR]

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(a) if $A \in \overrightarrow{BC}$ then $A \in \overrightarrow{BC}$ Contrapositive: if $A \notin \overrightarrow{BC}$ then $A \notin \overrightarrow{BC}$ Converse: if $A \in \overrightarrow{BC}$ then $A \in \overrightarrow{BC}$ (b) $\forall_x \forall_y \forall_z$ if $x = y$ then $xz = yz$ Contrapositive: $\forall_x \forall_y \forall_z$ if $xz \neq yz$ then $x \neq y$ Converse: $\forall_x \forall_y \forall_z$ if $xz = yz$ then $x = y$ (c) $\forall_x \forall_y$ if $x \neq 0$ then $xy \neq 0$ Contrapositive: $\forall_x \forall_y$ if $xy = 0$ then $x = 0$ Converse: $\forall_x \forall_y$ if $xy \neq 0$ then $x \neq 0$

An invalid inference is called a fallacy. On your work sheet write 'fallacy' for each of the invalid inferences.

(a) $\frac{a \neq b}{\text{if } a = b \text{ then } ac = bc}$
 $ac \neq bc$

(b) $\frac{a = b}{\text{if } a = b \text{ then } ac = bc}$
 $ac = bc$

(c) $\frac{a \neq b}{\text{if } ac = bc \text{ then } a = b}$
 $ac \neq bc$

(d) $\frac{\text{if } a = b \text{ then } ac = bc}{\text{if } a \neq b \text{ then } ac \neq bc}$

[RR]

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(a) and (d) are fallacies.

* * *

Consider the sentence:

(1) All cows are white.

What is a contradictory of (1). Since (1) is false, [We know of at least one cow that is not white], any sentence which is the contradictory of (1) must be true.

Consider the sentence:

(2) All cows are not white.

Is (2) true? No, (2) is false. We know of one cow that is white. So, (2) is not a contradictory of (1). A contradictory of

(1) All cows are white.

is

(3) Not all cows are white.

On your work sheet, write a contradictory of each of these statements.

(a) All numbers are prime.

(b) All even numbers are odd.

(c) All Gloxians have green hair.

(d) Not all birds are black.

[RR]

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- (a) Not all numbers are prime.
- (b) Not all even numbers are odd.
- (c) Not all Gloxians have green hair.
- (d) All birds are black.

* * *

For a contradictory of:

All numbers are prime.

you may have written:

- (1) Some numbers are not prime.

instead of:

- (2) Not all numbers are prime.

Since most people mean the same thing by (1) and (2) we will use sentences like (1) as well as sentences like (2) for contradictories of universal statements.

Use 'Some' and write a contradictory of each universal statement.

- (a) All numbers are prime.
- (b) All even numbers are odd.
- (c) All Gloxians have green hair.
- (d) All Gloxians do not have green hair.

[RR]

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- (a) Some numbers are not prime.
- (b) Some even numbers are not odd.
- (c) Some Gloxians do not have green hair.
- (d) Some Gloxians have green hair.

Consider the general statements:

- (1) All numbers are even
- (2) All numbers are not even

Translate statements (1) and (2) into universal generalizations. Use ' \forall '.

{RR}

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(1) \forall_x x is even(2) \forall_x x is not even

* * *

Since both (1) and (2) are false, (2) is not a contradictory of (1). Since

(1) \forall_x x is even

is false if;

(3) There is at least one number which is not even

is true, (3) is a contradictory of (1).

A translation of (3) into our formal language is:

(4) \exists_x x is not even

Hence, (4) is a contradictory of (1).

For each universal generalization, write a contradictory statement on your work sheet. Use ' \exists '.

(a) \forall_x $2x + 1 = 3x$ (b) \forall_x $x + 0 = x$ (c) \forall_x $x + 1 = x$

[RR]

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$$(a) \exists_x 2x + 1 \neq 3x$$

$$(b) \exists_x x + 0 \neq x$$

$$(c) \exists_x x + 1 \neq x$$

* * *

Note that a contradictory of a universal generalization:

$$(*) \forall_x 2 \cdot x + 1 = 3 \cdot x$$

is an existential generalization:

$$(**) \exists_x 2 \cdot x + 1 \neq 3 \cdot x$$

The existential generalization (**) is true since there does exist a number x such that $2 \cdot x + 1 \neq 3 \cdot x$. For example,

$$(A) 2 \cdot 5 + 1 \neq 3 \cdot 5$$

Sentence (A) is a counter-instance of the generalization (*) since A is true and (A) is a contradictory of an instance of (*).

Which of these sentences given below is a counter-instance of the generalization ' $\forall_x 2x + 1 = 3x$ '?

$$(a) 2 \cdot 1 + 1 = 3 \cdot 1$$

$$(b) 2 \cdot 8 + 2 \neq 3 \cdot 8$$

$$(c) 2 \cdot 7 + 1 \neq 3 \cdot 7$$

[RR]

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' $2 \cdot 7 + 1 \neq 3 \cdot 7$ ' is a counter-instance of the generalization ' $\forall_x 2x + 1 = 3x$ '.

* * *

Decide which of these sentences are true.

- (a) ' $2 \cdot 1 + 1 = 3 \cdot 1$ ' is true and ' $2 \cdot 1 + 1 = 3 \cdot 1$ ' is a contradictory of an instance of ' $\forall_x 2x + 1 = 3x$ '.
- (b) ' $2 \cdot 1 + 1 = 3 \cdot 1$ ' is not true and ' $2 \cdot 1 + 1 = 3 \cdot 1$ ' is not a contradictory of an instance of ' $\forall_x 2x + 1 = 3x$ '.
- (c) ' $2 \cdot 1 + 1 = 3 \cdot 1$ ' is not true or ' $2 \cdot 1 + 1 = 3 \cdot 1$ ' is not a contradictory of an instance of ' $\forall_x 2x + 1 = 3x$ '.

[RR]

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False (a) ' $2 \cdot 1 + 1 = 3 \cdot 1$ ' is true and ' $2 \cdot 1 + 1 = 3 \cdot 1$ ' is a contradictory of an instance of ' $\forall_x 2x + 1 = 3x$ '.

False (b) ' $2 \cdot 1 + 1 = 3 \cdot 1$ ' is not true and ' $2 \cdot 1 + 1 = 3 \cdot 1$ ' is not a contradictory of an instance of ' $\forall_x 2x + 1 = 3x$ '.

True (c) ' $2 \cdot 1 + 1 = 3 \cdot 1$ ' is not true or ' $2 \cdot 1 + 1 = 3 \cdot 1$ ' is not a contradictory of an instance of ' $\forall_x 2x + 1 = 3x$ '.

* * *

Decide which of these sentences are true.

- (a) $5 > 3$ and $2 + 1 = 4$
- (b) $5 \not> 3$ and $2 + 1 \neq 4$
- (c) $5 \not> 3$ or $2 + 1 \neq 4$
- (d) Ames is in Iowa and $5 \cdot 0 \neq 5$
- (e) Ames is not in Iowa and $5 \cdot 0 = 5$
- (f) Ames is not in Iowa or $5 \cdot 0 = 5$

[RR]

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- False (a) $5 > 3$ and $2 + 1 = 4$
- False (b) $5 \neq 3$ and $2 + 1 \neq 4$
- True (c) $5 \neq 3$ or $2 + 1 \neq 4$
- True (d) Ames is in Iowa and $5 \cdot 0 \neq 5$
- False (e) Ames is not in Iowa and $5 \cdot 0 = 5$
- False (f) Ames is not in Iowa or $5 \cdot 0 = 5$

* * *

On your work sheet, complete the table.

(a) $5 > 3$	True
$2 + 4 = 6$	True
$5 > 3$ <u>and</u> $2 + 4 = 6$	True
(b) $7 \cdot 9 = 9 \cdot 7$?
$3 + 1 = 3$?
$7 \cdot 9 = 9 \cdot 7$ <u>and</u> $3 + 1 = 3$?
(c) $8 + 2 \neq 2 + 8$?
$6 < 10$?
$8 + 2 \neq 2 + 8$ <u>and</u> $6 < 10$?
(d) Ames is not in Iowa	?
$5 \cdot 0 = 5$?
Ames is not in Iowa <u>and</u> $5 \cdot 0 = 5$?

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(a)	$5 > 3$	True
	$2 + 4 = 6$	True
	$5 > 3$ <u>and</u> $2 + 4 = 6$	True
(b)	$7 \cdot 9 = 9 \cdot 7$	True
	$3 + 1 = 3$	False
	$7 \cdot 9 = 9 \cdot 7$ <u>and</u> $3 + 1 = 3$	False
(c)	$8 + 2 \neq 2 + 8$	False
	$6 < 10$	True
	$8 + 2 \neq 2 + 8$ <u>and</u> $6 < 10$	False
(d)	Ames is not in Iowa	False
	$5 \cdot 0 = 5$	False
	Ames is not in Iowa <u>and</u> $5 \cdot 0 = 5$	False

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[RR]

[Page 211]

Sentences like:

$$5 > 3 \text{ and } 2 + 4 = 6$$

$$7 \cdot 9 = 9 \cdot 7 \text{ and } 3 + 1 = 3$$

$$8 + 2 \neq 2 + 8 \text{ and } 6 < 10$$

$$\text{Ames is not in Iowa and } 5 \cdot 0 = 0$$

are called conjunctions. In general, any sentence which is the result of substituting sentences for 'p' and 'q' in the expression:

p and q

is a conjunction.

Complete on your work sheet.

- (a) If both parts of a conjunction are true, the conjunction is .
(true/false)
- (b) If at least one part of a conjunction is false, then the conjunction is .
(true/false)
- (c) If both parts of a conjunction are false, the conjunction is .
(true/false)

[RR]

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- (a) If both parts of a conjunction are true, the conjunction is true.
- (b) If at least one part of a conjunction is false, then the conjunction is false.
- (c) If both parts of a conjunction are false, the conjunction is false.

* * *

The sentence:

$5 + 3 = 2$ or John is tall

is called an alternation. Any sentence which is the result of substituting sentences for 'p' and 'q' in the expression:

p or q

is an alternation.

Consider these two sentences:

- (1) John is sick
- (2) John is absent

Write, on your work sheet

- (a) a conjunction using (1) and (2)
- (b) an alternation using (1) and (2)

[RR]

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(a) John is sick and John is absent.

OR:

John is absent and John is sick.

(b) John is sick or John is absent.

OR:

John is absent or John is sick.

* * *

Decide which of these alternations are true. Indicate your decision by writing the appropriate letters on your work sheet.

(a) $5 > 3$ or $2 + 1 = 4$

(b) Ames is in Iowa or $5 \cdot 0 \neq 5$

(c) $7 \cdot 9 = 9 + 7$ or $3 \cdot 1 = 5$

(d) $5 + 3 = 7$ or $3 \cdot 1 = 3$

(e) $7 \cdot 9 = 9 \cdot 7$ or $8 + 3 = 11$

(f) Ames is not in Iowa or $5 \cdot 0 = 5$

[RR]

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a, b, d, and e are true.

In everyday language the word 'or' is used two ways.

(1) Suppose your father says to you "You may go to the dance or you may go to the football game". You probably take this to mean you may go to one of the activities but not both.

(2) However, a sign in a bookstore says:

A 10% discount is given to students or teachers.

Would the discount be denied to someone who is both a student and a teacher?
We believe not.

The first of these uses of 'or' is called the **exclusive** use. The second is the **inclusive** use. In mathematics we always use 'or' in the **inclusive** way.

On your work sheet, complete the table.

(a)	$6 + 2 = 8$	True
	$9 \cdot 7 = 7 \cdot 9$	True
	$6 + 2 = 8$ or $9 \cdot 7 = 7 \cdot 9$	True
(b)	$5 > 3$?
	$7 < 2$?
	$5 > 3$ or $7 < 2$?
(c)	$-(3 + 2) = -3 + 2$?
	$-(3 + 2) = -5$?
	$-(3 + 2) = -3 + 2$ or $-(3 + 2) = -5$?
(d)	$7 < 2$?
	$9 + 1 = 12$?
	$7 < 2$ or $9 + 1 = 12$?

[RR]

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(a)	$6 + 2 = 8$	True
	$9 \cdot 7 = 7 \cdot 9$	True
	$6 + 2 = 8$ or $9 \cdot 7 = 7 \cdot 9$	True
(b)	$5 > 3$	True
	$7 < 2$	False
	$5 > 3$ or $7 < 2$	True
(c)	$-(3 + 2) = -3 + 2$	False
	$-(3 + 2) = -5$	True
	$-(3 + 2) = -3 + 2$ or $-(3 + 2) = -5$	True
(d)	$7 < 2$	False
	$9 + 1 = 12$	False
	$7 < 2$ or $9 + 1 = 12$	False

* * *

Complete on your work sheet.

(a) If both parts of an alternation are true, then the alternation is .
(true/false)(b) If at least one part of an alternation is true, then the alternation is .
(true/false)(c) If both parts of an alternation are false, the alternation is .
(true/false)

[RR]

[Page 216]

- (a) If both parts of an alternation are true, then the alternation is true.
- (b) If at least one part of an alternation is true, then the alternation is true.
- (c) If both parts of an alternation are false, the alternation is false.

* * *

While talking with some of his classmates Pete said "All Gloxians eat meat and all Gloxians have green hair". Ann said, "That is not true". Suppose Ann is right. What can you conclude?

On your work sheet, write the sentence (or sentences) which follows logically from:

not (All Gloxians eat meat and all Gloxians have green hair)

- (a) Some Gloxians do not eat meat.
- (b) Some Gloxians do not have green hair.
- (c) Some Gloxians do not eat meat or some Gloxians do not have green hair.
- (d) Some Gloxians do not eat meat and some Gloxians do not have green hair.

[RR]

[Page 217]

The sentence:

Some Gloxians do not eat meat or some Gloxians do not have green hair.

is the only one of the sentences (a) - (d) which follows logically from:

not (All Gloxians eat meat and all Gloxians have green hair)

* * *

Pete had formed a club at Zabbranchburg. In order to be a member of the club a person must be a Zabbranchburg student and must be 16 years old. Harry could not be a member of the club.

What can you conclude about Harry?

Write your answer on your work sheet.

[RR]

[Page 218]

Harry is not a Zabbranchburg student or Harry is not 16 years old.

* * *

Several pairs of sentences are given below. In some pairs the second sentence follows logically from the first. On your work sheet, write the pairs of sentences in which the second follows from the first:

- (a) not(Sam is short and Mary is pretty)
Sam is not short or Mary is not pretty

- (b) not($a > 2$ and $a + 2 = 3$)
 $a \neq 2$ and $a + 2 \neq 3$

- (c) not($a + 2 \neq 3$ and $a \neq 1$)
 $a + 2 = 3$ or $a = 1$

- (d) $b - 3 \neq 5$ or $a + 7 \neq 13$
not($b - 3 = 5$ and $a + 7 = 13$)

(a) $\text{not}(\text{Sam is short and Mary is pretty})$
 $\text{Sam is not short or Mary is not pretty}$

(c) $\text{not}(a + 2 \neq 3 \text{ and } a \neq 1)$
 $a + 2 = 3 \text{ or } a = 1$

(d) $b - 3 \neq 5 \text{ or } a + 7 \neq 13$
 $\text{not}(b - 3 = 5 \text{ and } a + 7 = 13)$

* * *

We can summarize the results of our investigation of the denials of a conjunction by:

$$(I) \frac{\text{not}(p \text{ and } q)}{\text{not } p \text{ or not } q}$$

$$(II) \frac{\text{not } p \text{ or not } q}{\text{not}(p \text{ and } q)}$$

are valid inference patterns.

Use the inference pattern (I) and complete each inference.

(a) $\frac{\text{not}(a + 3 = 7 \text{ and } b = 8)}{?}$

(b) $\frac{\text{not}(\text{John is happy and John is poor})}{?}$

(c) $\frac{\text{not}(\text{Sue is tall and Harry is short})}{?}$

(d) $\frac{\text{not}(3 + a \neq 9 \text{ and } b - 3 = 7)}{?}$

[RR]

[Page 220]

- (a)
$$\frac{\text{not } (a + 3 = 7 \text{ and } b = 8)}{a + 3 \neq 7 \text{ or } b \neq 8}$$
- (b)
$$\frac{\text{not } (\text{John is happy and John is poor})}{\text{John is not happy or John is not poor.}}$$
- (c)
$$\frac{\text{not } (\text{Sue is tall and Harry is short})}{\text{Sue is not tall or Harry is not short.}}$$
- (d)
$$\frac{\text{not } (3 + a \neq 9 \text{ and } b - 3 = 7)}{3 + a = 9 \text{ or } b - 3 \neq 7}$$

To be technically correct we should have written 'not ($a + 3 = 7$) or not ($b = 8$)' as the answer for exercise (a). However, we have agreed that we may write ' $a + 3 \neq 7$ ' for ' $\text{not } (a + 3 = 7)$ '.

* * *

We have now learned about the denial of a conjunction. What is the denial of an alternation?

Pete told his classmates:

I guess I was wrong about the Gloxians. But I do know this:

All Gloxians eat bread or all Gloxians drink milk.

Ann said, "Pete, you're wrong again." Ann is right. What can you conclude?

On your work sheet, write the sentences which follow logically from:

not (all Gloxians eat bread or all Gloxians drink milk)

- (a) Some Gloxians do not eat bread
- (b) Some Gloxians do not drink milk
- (c) Some Gloxians do not eat bread and some Gloxians do not drink milk

[RR]

[Page 221]

- (a) Gloxians do not eat bread
- (b) Gloxians do not drink milk
- (c) Gloxians do not eat bread and Gloxians do not drink milk

* * *

Ann had also formed a club at Zabbranchburg. Everyone who is 15 years old or is in a math class is in Ann's club. Tom is not a member of Ann's club. What can you conclude about Tom?

On your work sheet, write ~~one~~ sentence which tells ~~everything~~ you can conclude about Tom from the information given above.

[RR]

[Page 222]

Tom is not 15 years old and Tom is not in a math class.

If you wrote only 'Tom is not 15 years old' or only 'Tom is not in a math class' you did not write **everything** you could conclude about Tom.

* * *

Complete each of these inferences.

(a) not ($a + 3 = 7$ or $b = 8$)

?

(b) not (John is happy or John is poor)

?

(c) not (Sue is tall or Harry is short)

?

(d) not ($3 + a \neq 9$ or $b - 3 = 7$)

?

[RR]

[Page 223]

- (a) $\frac{\text{not}(a + 3 = 7 \text{ or } b = 8)}{a + 3 \neq 7 \text{ and } b \neq 8}$
- (b) $\frac{\text{not}(\text{John is happy or John is poor})}{\text{John is not happy and John is not poor.}}$
- (c) $\frac{\text{not}(\text{Sue is tall or Harry is short})}{\text{Sue is not tall and Harry is not short.}}$
- (d) $\frac{\text{not}(3 + a \neq 9 \text{ or } b - 3 = 7)}{3 + a = 9 \text{ and } b - 3 \neq 7}$

* * *

We can summarize our investigation of the denial of an alternation by writing:

$$(I) \quad \frac{\text{not}(p \text{ or } q)}{\text{not } p \text{ and not } q}$$

$$(II) \quad \frac{\text{not } p \text{ and not } q}{\text{not}(p \text{ or } q)}$$

are valid inference patterns.

Turn to PAGE 224.

[RR]

[Page 224]

We have discussed rules of inference for universal generalizations, equality sentences, and conditionals. How are conjunctions and alternations used in reasoning?

Decide which of these inferences are correct. On your work sheet, copy the valid inferences.

$$(a) \frac{8 > 3 \text{ and } 5 + 1 = 6}{8 > 3}$$

$$(b) \frac{8 > 3 \text{ and } 7 + 1 = 11}{7 + 1 = 11}$$

$$(c) \frac{\text{John is absent}}{\text{John is absent and John is sick}}$$

$$(d) \frac{5 > 2 \quad \text{John is absent}}{5 > 2 \text{ and John is absent}}$$

Turn to PAGE 225a.

{Page 225a}

{RR}

$$(a) \frac{8 > 3 \text{ and } 5 + 1 = 6}{8 > 3}$$

$$(b) \frac{8 > 3 \text{ and } 7 + 1 = 11}{7 + 1 = 11}$$

$$(d) \frac{5 > 2 \quad \text{John is absent}}{5 > 2 \text{ and John is absent}}$$

Recall that the validity of an inference is independent of the truth or falsity of the premisses.

* * *

Here are three rules of inference which summarize the way people use conjunctions.

(I) $\frac{p \quad q}{p \text{ and } q}$	(II) $\frac{p \text{ and } q}{p}$	(III) $\frac{p \text{ and } q}{q}$
--	-----------------------------------	------------------------------------

Hence, if you want to derive a conjunction from given premisses, all you need to do is derive each of its two parts from the premisses and then use an inference of the first kind. Here, for example, is a derivation of the sentence:

$$\forall_x \forall_y \forall_z \text{ if } x = y \text{ then } x + z = y + z \text{ and } x - z = y - z$$

from the premisses:

$$\forall_x \forall_y \forall_z \text{ if } x = y \text{ then } x + z = y + z$$

$$\forall_x \forall_y x - y = x + -y$$

Turn to PAGE 55.

SKIP THIS PAGE

[RR]

[Page 225]

- (1) $a = b$ Assumption *
-
- (2) if $a = b$ then $a + c = b + c$ $\forall_x \forall_y \forall_z$ if $x = y$ then $x + z = y + z$ UI
-
- (3) $a + c = b + c$ (1), (2); MP
-
- (4) if $a = b$ then $a + -c = b + -c$ $\forall_x \forall_y \forall_z$ if $x = y$ then $x + z = y + z$ UI
-
- (5) $a + -c = b + -c$ (1), (4); MP
-
- (6) $a - c = a + -c$ $\forall_x \forall_y$ $x - y = x + -y$ UI
-
- (7) $a - c = b + -c$ (6), (5); RRE
-
- (8) $b - c = b + -c$ $\forall_x \forall_y$ $x - y = x + -y$ UI
-
- (9) $a - c = b - c$ (8), (7); RRE'
-
- (10) $a + c = b + c$ and $a - c = b - c$ (3), (9); $\frac{p}{p \text{ and } q}$
-
- (11) if $a = b$ then $a + c = b + c$ and $a - c = b - c$ C(10), D(1)*
-
- (12) $\forall_x \forall_y \forall_z$ if $x = y$ then $x + z = y + z$ and $x - y = x - z$ (1) - (11); TPP

In your later work in mathematics you will have an opportunity to use these rules of inference for conjunctions in proving many theorems. It is our intention here to only introduce you to these rules and you should not expect to obtain complete mastery of them.

Turn to PAGE 226.

[RR]

[Page 226]

Here are two basic principles about '>'.

$$(a) \quad \forall_x \forall_y \forall_z \text{ if } x > y \text{ then } x + z > y + z$$

$$(b) \quad \forall_x \forall_y \forall_z \text{ if } x > y \text{ and } y > z \text{ then } x > z$$

Suppose we want to derive the sentence:

$$(*) \quad \forall_x \forall_y \forall_u \forall_v \text{ if } (x > y \text{ and } u > v) \text{ then } x + u > y + v$$

On your work sheet, complete these sentences about a derivation of (*).

1. A pattern sentence we need to derive as the next to the last step in the derivation is 'if ($a > b$ and $c > d$) then _____?_____ '.
2. Since the pattern sentence is a conditional, we could use _____?,
(the antecedent of the pattern sentence) as an assumption.
3. In order to use an instance of (b) to derive the sentence ' $a + c > b + d$ ',
we could establish the conjunction ' $a + c > b + c$ and _____?_____ '.
4. The rule of reasoning which would justify inferring the consequent of an
instance of (b) from the instance together with the antecedent of the instance
is _____?_____.

[RR]

[Page 227]

1. A pattern sentence we need to derive as the next to the last step in the derivation is 'if($a > b$ and $c > d$) then $a + c > b + d$ '.
2. Since the pattern sentence is a conditional, we could use ' $a > b$ and $c > d$ ', the antecedent of the pattern sentence, as an assumption.
3. In order to use an instance of (b) to derive the sentence ' $a + c > b + d$ ', we could establish the conjunction ' $a + c > b + c$ and $b + c > b + d$ '.
4. The rule of reasoning which would justify inferring the consequent of an instance of (b) from the instance together with the antecedent of the instance is modus ponens.

Turn to PAGE 228.

[RR]

[Page 228]

Suppose someone tells you that Harry is sick or Joan is pretty. Is it correct for you to conclude that Harry is sick? The answer is 'No'. Can you conclude that Joan is pretty? Again the answer is 'No'. The alternation:

Harry is sick or Joan is pretty

actually gives you no information about Harry and no information about Joan. However, if you find out that Joan is not pretty, you may then use the two premisses:

Harry is sick or Joan is pretty

Joan is not pretty

and conclude:

Harry is sick

Decide which of the inferences given below are valid. On your work sheet write the valid inferences.

- (a) Pete is tall or Sam is short
Sam is short
- (b) Pete is tall and Sam is short
Sam is short
- (c) Pete is tall or Sam is short Pete is not tall
Sam is short
- (d) Pete is tall or Sam is short Sam is short
Pete is not tall
- (e) Harry is absent or Jane is here Jane is not here
Harry is absent

[RR]

[Page 229]

(b) $\frac{\text{Pete is tall and Sam is short}}{\text{Sam is short}}$

(c) $\frac{\text{Pete is tall or Sam is short}}{\text{Sam is short}} \quad \frac{\text{Pete is not tall}}{\text{Sam is short}}$

(e) $\frac{\text{Harry is absent or Jane is here}}{\text{Harry is absent}} \quad \frac{\text{Jane is not here}}{\text{Harry is absent}}$

* * *

Recall the club that Ann had started at Zabbranchburg. (1) If a person is 15 years old or is in a math class, then that person is in the club. Furthermore (2) if a person is in the club then he is 15 years old or is in a math class.

In each example given below, tell what you can conclude from sentences (1) and (2) and the information given in the example.

(a) Joan is in the club

(b) Harry is 15 years old

(c) Sam is 15 years old and Sam is in a math class

(d) Sue is not in the club

(e) Pete is in the club and Pete is not 15 years old

(f) Lois is 15 years old or Lois is in a math class

* * *

- (1) hand in a messy paper
- (2) not hand in a paper

On your work sheet, complete each inference.

- (a) $a > 0$ or $-a > 0$ if $a > 0$ then $a^2 > 0$ if $-a > 0$ then $a^2 > 0$
- (b) Tom is 15 or Tom is tall if Tom is 15 then Tom is happy if ? then ?
Tom is happy
- (c) ? or ? if $a \neq 1$ then $a^2 \neq a$ if $a \neq 0$ then $a^2 \neq a$
 $a^2 \neq a$

[RR]

[Page 231]

- (a)
$$\frac{a > 0 \text{ or } -a > 0 \quad \text{if } a > 0 \text{ then } a^2 > 0 \quad \text{if } -a > 0 \text{ then } a^2 > 0}{a^2 > 0}$$
- (b)
$$\frac{\text{Tom is 15 or Tom is tall} \quad \begin{array}{l} \text{if Tom is 15} \\ \text{then Tom is happy} \end{array} \quad \begin{array}{l} \text{if Tom is tall} \\ \text{then Tom is happy} \end{array}}{\text{Tom is happy}}$$
- (c)
$$\frac{a \neq 1 \text{ or } a \neq 0 \quad \text{if } a \neq 1 \text{ then } a^2 \neq a \quad \text{if } a \neq 0 \text{ then } a^2 \neq a}{a^2 \neq a}$$

* * *

Each of the inferences (a), (b), and (c) is justified by a rule of reasoning for alternations. This rule is:

The Rule of Dilemma [RD]

$$\frac{p \text{ or } q \quad \text{if } p \text{ then } r \quad \text{if } q \text{ then } r}{r}$$

is a valid inference pattern.

On your work sheet, complete these sentences about a derivation of:

$$\forall_x \forall_y \text{ if } x = 0 \text{ or } y = 0 \text{ then } x \cdot y = 0$$

- (i) In the pattern sentence 'if $a = 0$ or $b = 0$ then $a \cdot b = 0$ ' the consequent is (1) and the antecedent is (2).
- (ii) In order to use The Rule of Dilemma we need to derive the two sentences, 'if $a = 0$ then $a \cdot b = 0$ ' and 'if (3) then $a \cdot b = 0$ '.
- (iii) The assumption could then be discharged to get the pattern sentence '(4)'.

[RR]

[Page 232]

(1) $a \cdot b = 0$

(3) if $b = 0$ then $a \cdot b = 0$

(2) $a = 0$ or $b = 0$

(4) if $a = 0$ or $b = 0$ then $a \cdot b = 0$

* * *

We can derive the sentence ' $3 + -3 = 0$ '. [From which principle?] Can we derive the sentence

(1) $(3 + -3) \cdot 5 = 0$

from the premisses:

(2) $3 + -3 = 0$

(3) $\forall_x \forall_y$ if $x = 0$ or $y = 0$ then $xy = 0$?

Let's consider an instance of (3).

(4) if $3 + -3 = 0$ or $5 = 0$ then $(3 + -3) \cdot 5 = 0$

The consequent of (4) [$(3 + -3) \cdot 5 = 0$] is what we want. Hence, if we can get the antecedent [$3 + -3 = 0$ or $5 = 0$] then we can use modus ponens to get what we want. We do not have the alternation:

(5) $3 + -3 = 0$ or $5 = 0$

but we do have the sentence:

(2) $3 + -3 = 0$

Does the alternation (5) follow logically from (2)? The answer is 'Yes'. In fact, the result of writing any sentence after 'or' in:

$3 + -3 = 0$ or

follows logically from:

$3 + -3 = 0$

On your work sheet, write a complete derivation of the sentence:

$(3 + -3) \cdot 5 = 0$

Use only principles or theorems as premisses.

[RR]

[Page 233]

$$\begin{array}{c}
 \frac{\forall x \quad x + -x = 0}{3 + -3 = 0} \text{ UI} \\
 \frac{3 + -3 = 0 \text{ or } 5 = 0}{(3 + -3) \cdot 5 = 0} \text{ ?}
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{\forall x \forall y \quad \text{if } x = 0 \text{ or } y = 0 \text{ then } xy = 0}{\text{if } 3 + -3 = 0 \text{ or } 5 = 0 \text{ then } (3 + -3) \cdot 5 = 0} \text{ UI} \\
 (3 + -3) \cdot 5 = 0 \text{ MP}
 \end{array}$$

* * *

Here are valid inference patterns for alternations.

$$(I) \quad \frac{p}{p \text{ or } q}$$

$$(II) \quad \frac{q}{p \text{ or } q}$$

$$(III) \quad \frac{p \text{ or } q \quad \text{not } p}{q}$$

$$(IV) \quad \frac{p \text{ or } q \quad \text{not } q}{p}$$

Compare these with the inference patterns for conjunctions. Inference pattern (I) was used in the derivation above.

Turn to PAGE 234.

[RR]

[Page 234]

Pete had found out from the Gloxians that the sentence:

$$\forall x \forall y \text{ if } x = 0 \text{ or } y = 0 \text{ then } x \cdot y = 0$$

and its converse:

$$\forall x \forall y \text{ if } x \cdot y = 0 \text{ then } x = 0 \text{ or } y = 0$$

were theorems. He also knew that the product of the Gloxian numbers \square and Δ is 0. Pete told Ann and Lois about these theorems and also told Ann:

$$\square \cdot \Delta = 0$$

What can Ann conclude? Write your answer on your work sheet.

Pete told Lois:

$$\square \cdot \odot \neq 0$$

What can Lois conclude? Write your answer on your work sheet.

Pete knows:

$$\square \cdot \Delta = 0$$

$$\square \cdot \odot \neq 0$$

What can Pete conclude? Write your answer on your work sheet.

[RR]

[Page 235]

Ann: $\square = 0$ or $\Delta = 0$

Lois: $\square \neq 0$ and $\odot \neq 0$

Pete: $\Delta = 0$

* * *

When questioned about her reasoning Lois answered:

From an instance of ' $\forall_x \forall_y$ if $x = 0$ or $y = 0$ then $xy = 0$ ' and what Pete told me ' $\square \neq 0$ ', I could get ' $\text{not}(\square = 0 \text{ or } \odot = 0)$ ' by modus tollens. And I also know that ' $\square \neq 0$ and $\odot \neq 0$ ' follows from ' $\text{not}(\square = 0 \text{ or } \odot = 0)$ '.

On your work sheet,

write a column derivation of the sentence ' $\square \neq 0$ and $\odot \neq 0$ '. Use Lois's answer as a guide.

[RR]

[Page 236]

	$\forall_x \forall_y$ if $x = 0$ or $y = 0$ then $xy = 0$	
if $\square = 0$ or $\odot = 0$ then $\square \cdot \odot = 0$		UI
$\square \cdot \odot \neq 0$	Assumption	
$\text{not}(\square = 0 \text{ or } \odot = 0)$		(2), (1); MT
$\square \neq 0 \text{ and } \odot \neq 0$		(3); $\frac{\text{not}(p \text{ or } q)}{\text{not } p \text{ and not } q}$

* * *

Justify Pete's conclusion by writing a complete derivation of the sentence:

$$\forall_x \forall_y \forall_z \text{ if } xy = 0 \text{ and } xz \neq 0 \text{ then } y = 0$$

[You may use the theorems ' $\forall_x \forall_y$ if $xy = 0$ then $x = 0$ or $y = 0$ ' and ' $\forall_x \forall_y$ if $x = 0$ or $y = 0$ then $xy = 0$ ' as premisses.]

[RR]

[Page 237]

1)	$ab = 0$ and $ac \neq 0$	Assumption *
2)	$ab = 0$	(1); $\frac{p \text{ and } q}{p}$
	$\forall x \forall y$ if $xy = 0$ then $x = 0$ or $y = 0$	
3)	if $ab = 0$ then $a = 0$ or $b = 0$	UI
4)	$a = 0$ or $b = 0$	(2), (3); MP
	$\forall x \forall y$ if $x = 0$ or $y = 0$ then $xy = 0$	
5)	if $a = 0$ or $c = 0$ then $ac = 0$	UI
6)	$ac \neq 0$	(1); $\frac{p \text{ and } q}{q}$
7)	$\text{not}(a = 0 \text{ or } c = 0)$	(6), (5); MT
8)	$a \neq 0$ and $c \neq 0$	(7); $\frac{\text{not}(p \text{ or } q)}{\text{not } p \text{ and not } q}$
9)	$a \neq 0$	(8); $\frac{p \text{ and } q}{p}$
10)	$b = 0$	(9), (4); $\frac{p \text{ or } q \text{ not } p}{q}$
11)	if $ab = 0$ and $ac \neq 0$ then $b = 0$	C(10); D(1) *
12)	$\forall x \forall y \forall z$ if $xy = 0$ and $xz \neq 0$ then $y = 0$	(1) - (11); TPP

You may have a derivation different from ours and still be right. If you have any doubts, check with your teacher.

Turn to PAGE 238.

[RR]

[Page 238]

We have shown that the sentence:

$$(1) \quad \forall_x \forall_y \text{ if } x + y = 0 \text{ then } x = -y$$

is a theorem.

Is the sentence:

$$(*) \quad \forall_x \forall_y \text{ if } x + y \neq 0 \text{ then } x \neq -y$$

a theorem? Let's prove that $(*)$ is a theorem. But how should we start?

We know that we want to derive the pattern sentence:

$$(1) \quad \text{if } a + b \neq 0 \text{ then } a \neq -b$$

We have seen that when we want to derive a conditional sentence, we can use the antecedent as a premiss. But the antecedent here ' $a + b \neq 0$ ' is a denial sentence and none of our principles are about denial sentences. There is a way out. Notice that

$$(1) \quad \text{if } a + b \neq 0 \text{ then } a \neq -b$$

is the contrapositive of:

$$(2) \quad \text{if } a = -b \text{ then } a + b = 0$$

So, let's derive (2) and then use contraposition to get (1).

On your work sheet, prove that the sentence:

$$\forall_x \forall_y \text{ if } x + y \neq 0 \text{ then } x \neq -y$$

is a theorem.

[RR]

[Page 239]

Here is our proof:

* $a = -b$	$\frac{\forall x \forall y \quad x + y = y + x}{b + -b = -b + b}$	UI	$\frac{\forall x \quad x + -x = 0}{b + -b = 0}$	UI
	$-b + b = 0$			RRE
	$a + b = 0$			RRE
	if $a = -b$ then $a + b = 0$			C, D *
	if $a + b \neq 0$ then $a \neq -b$			Cp
	$\forall x \forall y$ if $x + y \neq 0$ then $x \neq -y$			TPP

If you have a different proof, check with your teacher.

Turn to PAGE 240.

In this book we have discussed several rules of reasoning which are useful in proving theorems. You may wish to review these before continuing your study of the book. Here is a summary of the rules of reasoning we have discussed:

Universal Instantiation	[Page 68]
The Replacement Rule for Equations	[Page 75a]
Test Pattern Principle	[Page 86]
Law of Identity	[Page 106]
Modus Ponens	[Page 154]
Conditionalizing	[Page 156]
The Discharge Rule	[Page 159]
Double Denial	[Page 181]
Reverse Double Denial	[Page 181]
Modus Tollens	[Page 185]
Contraposition	[Page 197]
Denial of a Conjunction	[Page 219]
Denial of an Alternation	[Page 223]
Conjunctions	[Page 225a]
Rule of Dilemma	[Page 231]
Alternations	[Page 233]

Turn to PAGE 241.

[RR]

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The basic rules of inference can be used to derive other rules. For example, from the premisses:

- (i) if today is Tuesday then grass is green
- (ii) if grass is green then the corn is tall

we would conclude

- (iii) if today is Tuesday then the corn is tall.

Here is the inference pattern which is used to infer (iii) from (i) and (ii):

$$(I) \frac{\frac{\text{if } p \text{ then } q \quad \text{if } q \text{ then } r}{\text{if } p \text{ then } r}}$$

The inference pattern (I) can be derived as follows:

$$\begin{array}{c} * \\ \frac{\frac{p \quad \text{if } p \text{ then } q}{q} \quad \text{MP} \quad \text{if } q \text{ then } r}{r} \quad \text{MP} \\ \hline \text{if } p \text{ then } r \quad * [C, D] \end{array}$$

(I) is called the hypothetical syllogism.

Here is another inference pattern which can be derived from the basic rules of inference:

Importation

$$\frac{\text{if } p \text{ then } [\text{if } q \text{ then } r]}{\text{if } [p \text{ and } q] \text{ then } r}$$

Here is the start of a derivation. You complete it.

$$\begin{array}{c} * \\ \frac{\frac{p \text{ and } q}{q} \quad \frac{\frac{p \text{ and } q}{p} \quad \text{if } p \text{ then } [\text{if } q \text{ then } r]}{?}}{r} \quad * \\ \hline ? \end{array}$$

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$$\begin{array}{c}
 * \\
 \frac{\frac{p \text{ and } q}{q}}{\quad} \quad \frac{\frac{p \text{ and } q}{p} \quad \text{if } p \text{ then [if } q \text{ then } r]}{\text{if } q \text{ then } r} \\
 \hline
 r \\
 \hline
 \text{if } (p \text{ and } q) \text{ then } r \quad *
 \end{array}$$

* * *

An inference pattern closely related to importation:

$$\text{(I)} \quad \frac{\text{if } p \text{ then [if } q \text{ then } r]}{\text{if [} p \text{ and } q \text{] then } r}$$

is exportation:

$$\text{(II)} \quad \frac{\text{if [} p \text{ and } q \text{] then } r}{\text{if } p \text{ then [if } q \text{ then } r]}$$

Here is the start of a derivation of (II). You complete the derivation.

$$\begin{array}{c}
 + \quad * \\
 \frac{p \quad q}{p \text{ and } q} \quad \text{if } (p \text{ and } q) \text{ then } r \\
 \hline
 ? \\
 \hline
 \text{if } q \text{ then } r \quad * \\
 \hline
 ? \text{ then } (\quad ? \quad) \quad +
 \end{array}$$

[RR]

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+	*
p	q
p and q	if (p and q) then r
	r
	if q then r
	if p then (if q then r)
	*
	+

* * *

As you continue your study of mathematics you will use these inference schemes and others to prove theorems. We have not attempted to discuss all the types of reasoning which you will use. However, we hope that whenever you do use a type of reasoning which we have not discussed, you will be able to justify it on the basis of what you have learned.

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[RR]

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On validity and truth

Truth is a property of statements--some statements are true, some are not. Statements which are not true are called false. [Sentences which are not statements--that is, open sentences such as ' $a = 1$ '--are neither true nor false.]

Validity is a property of inferences. For example, the inference:

$$\begin{array}{ccc}
 & (1) & (2) \\
 (**) & \frac{\text{John is poor}}{\text{if John is poor then John is happy}} & \\
 & \text{John is happy} & (3)
 \end{array}$$

is valid. Recall that our justification for rating the inference (**) as valid comes merely from the way people use 'if...then...'. The validity of the inference (**) is a consequence solely of the fact that (2) is a conditional sentence, (1) is its antecedent, and (3) is its consequent. Which, if any, of the three statements are true, and which are false has no bearing on the validity of the inference.

Suppose we believe that the sentences:

- (1) John is poor
 and
 (2) if John is poor then John is happy

are true, and because of the validity of the inference (**), we accept the sentence:

- (3) John is happy

as true. Now, if we discover that (3) is false, we must not say that we have reasoned incorrectly, that is, that the inference is not valid. Rather, we should conclude that we were incorrect in believing that both (1) and (2) were true.

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[RR]

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Here are two inferences:

$$(I) \quad \frac{2 = 1 + 1 \quad 2 + 3 = 5}{(1 + 1) + 4 = 6}$$

$$(II) \quad \frac{1 = 2 \quad 2 = 1}{1 = 1}$$

Answer these questions on your work sheet.

(a) Are the premisses of inference (I) true?

(b) Is the conclusion of inference (I) true?

(c) Is inference (I) a valid inference?

(d) Are the premisses of inference (II) true?

(e) Is the conclusion of inference (II) true?

(f) Is inference (II) a valid inference?

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- (a) Yes (b) Yes (c) No (d) No (e) Yes (f) Yes

$$(I) \quad \begin{array}{c} \text{True} \quad \quad \text{True} \\ 2 = 1 + 1 \quad 2 + 3 = 5 \\ \hline \text{INVALID} \end{array}$$

$$(II) \quad \begin{array}{c} \text{False} \quad \quad \text{False} \\ 1 = 2 \quad 2 = 1 \\ \hline 1 = 1 \quad \text{VALID} \end{array}$$

We have seen (Inference I) that we can have true premisses and true conclusions and an invalid inference. Hence, the truth of premisses and conclusion does not guarantee the validity of an inference. We have also seen (Inference II) that the validity of an inference does not guarantee the truth of the premisses and conclusion.

However, the types of inference we have chosen to call valid are such that if

(a) an inference is valid

and

(b) the premisses are true

then

(c) the conclusion is true.

Decide which of these sentences are true. Write the true sentences on your work sheet.

- (a) When the premisses of an inference are true and the conclusion is true, it follows that the inference is valid.
- (b) When an inference is valid and the conclusion is true, it follows that the premisses are true.
- (c) When an inference is valid and the conclusion is false, it follows that at least one premiss is false.
- (d) When an inference is valid and the premisses are true, it follows that the conclusion is true.

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- (c) When an inference is valid and the conclusion is false, it follows that at least one premiss is false.
- (d) When an inference is valid and the premisses are true, it follows that the conclusion is true.

* * *

The validity of an inference does not guarantee the truth of its premisses, or of its conclusion. In fact, we often find the need to reason from a premiss which we believe to be false. Consider the conditional

$$(1) \text{ if } 3 \cdot 0 = 5 \cdot 0 \text{ then } 3 = 5$$

We believe (1) to be false, and can show that (1) is false in this manner:

$$\begin{array}{rcl} \frac{\forall_x x \cdot 0 = 0}{3 \cdot 0 = 0} & \frac{\forall_x x \cdot 0 = 0}{5 \cdot 0 = 0} & \\ \hline 3 \cdot 0 = 5 \cdot 0 & & \text{if } 3 \cdot 0 = 5 \cdot 0 \text{ then } 3 = 5 \\ \hline 3 = 5 & & \end{array}$$

Now, '3 = 5' is false and the inference is valid. Hence, at least one of the premisses is false. The premisses are:

$$\forall_x x \cdot 0 = 0$$

$$\text{if } 3 \cdot 0 = 5 \cdot 0 \text{ then } 3 = 5$$

Since the principle ' $\forall_x x \cdot 0 = 0$ ' is not false, the only other premiss, 'if $3 \cdot 0 = 5 \cdot 0$ then $3 = 5$ ' must be false.

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[RR]

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In showing that 'if $3 \cdot 0 = 5 \cdot 0$ then $3 = 5$ ' is false, we have used a general rule concerning truth and inferences. The rule is:

- (i) If some consequence of a set of premisses is false, and the inference is valid, then at least one of the premisses is false.

(i) is equivalent to:

- (i') If a sentence is a consequence of a valid inference whose only premisses are true, then the sentence is true.

We can use (i) and (i') together with our inference patterns for conjunctions to justify the decisions we made concerning when a conjunction is true and when it is false.

Here are the valid inference patterns for conjunctions.

CONJUNCTIONS

$$(I) \frac{p \text{ and } q}{p}$$

$$(II) \frac{p \text{ and } q}{q}$$

$$(III) \frac{p \quad q}{p \text{ and } q}$$

Suppose a statement $[p]$ is false. It follows from (I) and (i) that the conjunction $[p \text{ and } q]$ is false. Similarly, if a statement $[q]$ is false, the conjunction $[p \text{ and } q]$ is false. From (III) and (i'), whenever a statement $[p]$ is true and a statement $[q]$ is true, the conjunction $[p \text{ and } q]$ is true. Thus, we have the truth table for conjunctions.

p	q	p and q
T	T	T
T	F	F
F	T	F
F	F	F

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Note that the truth value of a conjunction is determined by the truth value of its components. In general,

- (ii) The truth value of a complex statement depends only on the truth value of its components.

We can also use the rules of inference for conditionals together with (i), (i'), and (ii) to get the truth table for conditionals. Here is an incomplete table:

CONDITIONALS

p	q	if p then q
T	T	?
T	F	?
F	T	?
F	F	

The inference patterns for conditionals which we will use are:

CONDITIONALS

$$(I) \frac{p \quad \text{if } p \text{ then } q}{q}$$

$$(II) \frac{q}{\text{if } p \text{ then } q}$$

$$(III) \frac{\begin{array}{c} * \\ p \\ \vdots \\ q \end{array}}{\text{if } p \text{ then } q} *$$

Recall that (I) is the inference pattern for modus ponens, (II) is the inference pattern for conditionalizing, and (III) is the pattern for discharging an assumption.

On your work sheet, make a truth table like the one above and complete the first three lines.

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p	q	if p then q
T	T	T
T	F	F
F	T	T
F	F	?

* * *

Here is our justification for lines (1) and (3).

Suppose a true statement is substituted for 'q' in the valid inference pattern:

$$\frac{q}{\text{if } p \text{ then } q}$$

Now, no matter what kind of statement, true or false, is substituted for 'p', the resulting statement of the form 'if p then q' is a consequence of a true statement. Hence, by (i'), any conditional with a true consequent is true

Here is our justification for line (2).

Suppose a true statement is substituted for 'p' in the inference pattern:

$$\frac{p \quad \text{if } p \text{ then } q}{q}$$

and a false statement is substituted for 'q'.

We now have a false statement [q] which is a consequence of the premisses [p] and [if p then q]. Hence, by (2), at least one of the premisses must be false. But [p] is true. Hence, [if p then q] must be false. This justifies the entry in line (2).

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In order to complete the fourth line in the truth table for conditionals we need to discuss the truth value of a sentence which is a consequence of no premisses. Here is such a sentence:

(**) If John is absent then John is absent.

and here is a derivation of (**):

			*
	*	John is absent	
John is absent		if John is absent then John is absent	C
		John is absent	MP
		if John is absent then John is absent	C, D*

What is the truth value of (**)? Could (**) be false?

Recall

- (i) if some consequence of a set of premisses is false,
and the inference is valid, then at least one of the
premisses is false

So, if (**) is false then by (i), at least one of the premisses must be false. But (**) is a consequence of the empty set of premisses. ['John is absent' has been discharged.] Since there are no premisses there can be no false premisses. Hence, (**) is always true. In particular, (**) is true even when 'John is absent' is false.

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When 'John is absent' is false, the conditional:

if John is absent then John is absent

is a conditional whose antecedent and consequent are both false. Now, by

- (ii) The truth value of a complex statement depends
only on the truth value of its components.

it follows that since (**) is true, each conditional whose antecedent and consequent are both false, must, also, be true.

Thus, we complete the truth table for conditionals.

CONDITIONALS

p	q	if p then q
T	T	T
T	F	F
F	T	T
F	F	T

Suppose someone says:

If you jump that fence, I'll eat my hat.

and you do not jump the fence and he does not eat his hat. You would still say that he had made a true statement.

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[RR]

[Page 252]

We can use the general rules concerning truth together with our rules of inference to justify truth tables for denial sentences and alternation sentences. These truth tables are:

DENIALS

p	not p
T	F
F	T

ALTERNATIONS

p	q	p or q
T	T	T
T	F	T
F	T	T
F	F	F

We actually need one more rule concerning truth. This rule is:

(iii) Not all statements are true.

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[RR]

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Recall our discussion of contradictory statements. Contradictory statements are two statements such that if either is true then the other must be false and at least one is true.

Consider the conditional:

(I) if John is sick then John is absent

What is a contradictory of (**)?

Here is a truth table for (I) and a related conditional.

(II) if John is not sick then John is absent

John is sick	John is absent	John is not sick	John if is then sick	John is absent	John if is then not sick	John is absent
--------------------	----------------------	------------------------	----------------------------	----------------------	--------------------------------	----------------------

p	q	not p	if p then q	if not p then q
T	T	F	T	T
T	F	F	?	T
F	T	?	?	?
F	F	?	?	?

Complete the table on your work sheet.

[RR]

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John is sick	John is absent	John is not sick	John if is then sick sick	John is absent	John if is then not sick not sick	John is absent
--------------------	----------------------	------------------------	---------------------------------	----------------------	---	----------------------

p	q	not p	if p then q	if not p then not q
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	F

* * *

From line 1 of the truth table we see that

(1) if John is not sick then John is absent

could not be a contradictory of

(**) if John is sick then John is absent

[Both (1) and (**) are true when 'John is sick' is true and 'John is absent' is true.]

In a similar way you could show that neither of the conditionals:

if John is sick then John is not absent

if John is not sick then John is not absent

could be a contradictory of:

(**) if John is sick then John is absent

Decide if any of these conjunctions could be a contradictory of (**). On your work sheet, write 'YES' or write 'NO' according to your decision.

(a) John is not sick and John is not absent.

(b) John is not sick and John is absent.

(c) John is sick and John is not absent.

[RR]

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(a) No

(b) No

(c) Yes

* * *

Here is a truth table which shows that:

John is sick and John is not absent

is a contradictory of:

if John is sick then John is absent.

John is sick	John is absent	John is not absent	if John is sick then John is absent	John is sick and John is not absent
--------------------	----------------------	--------------------------	--	---

p	q	not q	if p then q	p and not q
T	T	F	T	F
T	F	T	F	T
F	T	F	T	F
F	F	T	T	F

For each conditional, write, on your work sheet, a conjunction which is a contradictory of the conditional.

(a) If John is sick then John is absent (b) if Sue is pretty then Tom is smart

(c) if Pete is not tall then John is sick (d) if $a + 2 \neq 5$ then $a \neq 3$

(e) if $3 \cdot 0 = 5 \cdot 0$ then $3 = 5$

[RR]

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- (a) John is sick and John is not absent (b) Sue is pretty and Tom is not smart
 (c) Pete is not tall and John is not sick (d) $a + 2 \neq 5$ and $a = 3$
 (e) $3 \cdot 0 = 5 \cdot 0$ and $3 \neq 5$

* * *

The generalization:

$$\forall_x \forall_y \forall_z \text{ if } x = y \text{ then } xz = yz$$

is a theorem.

Is its converse:

$$(I) \quad \forall_x \forall_y \forall_z \text{ if } xz = yz \text{ then } x = y$$

a theorem?

Bruce had tried to derive (I) from the principles and was not successful. He suspected that (I) is not a theorem so he looked for a sentence which would show that (I) is not a theorem.

An instance of (I) is:

$$(*) \text{ if } 3 \cdot 0 = 5 \cdot 0 \text{ then } 3 = 5$$

The sentence:

$$3 \cdot 0 = 5 \cdot 0 \text{ and } 3 \neq 5$$

is true and is a contradictory of (*). Hence, (*), a consequence of (I), is false. So, we say that (I) is not a theorem. In general, if a given sentence has a false consequence, we say that the sentence is not a theorem in our system.

Turn to PAGE 257.

[RR]

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Here are two inference patterns which show that a sentence of the form:

p and not q

is equivalent to a sentence of the form:

not (if p then q)

$$\begin{array}{c}
 \text{(I)} \quad \frac{\frac{\frac{p \text{ and not } q}{p} \quad \frac{*}{\text{if } p \text{ then } q}}{q} \quad \frac{*}{\text{if } p \text{ then } q} \quad \frac{p \text{ and not } q}{\text{not } q}}{\text{not (if } p \text{ then } q)}
 \end{array}$$

$$\begin{array}{c}
 \text{(II)} \quad \frac{\frac{*}{\text{not (p and not q)}} \quad \frac{\frac{*}{\text{not p}} \quad \frac{\text{if not } q \text{ then not p}}{\text{if } p \text{ then } q}}{\text{if not } p \text{ then (if } p \text{ then } q)} \quad \frac{\frac{\dagger}{q} \quad \frac{\text{if } p \text{ then } q}{\text{if } q \text{ then (if } p \text{ then } q)}}{\dagger}}{\text{if } p \text{ then } q} \quad \frac{*}{\text{if [not (p and not q)] then [if } p \text{ then } q]}} \\
 \frac{\text{not (if } p \text{ then } q) \quad \text{if [not (if } p \text{ then } q)] \text{ then (p and not } q)}}{p \text{ and not } q}
 \end{array}$$

These inference patterns could be used to show that:

(1) not (if John is sick then John is absent)

(2) John is sick and John is not absent

are equivalent sentences.

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[RR]

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We have seen that:

- and
- (1) not (if John is sick then John is absent)
 - (2) John is sick and John is not absent

are equivalent sentences. Hence, the sentence:

- (3) not (John is sick and John is not absent)

can be translated to:


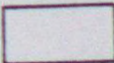
- (4) not [not (if John is sick then John is absent)]

Now, by reverse double denial,

- (5) if John is sick then John is absent

follows from (4). Since (5) follows from (4) and (4) follows from (3), (5) is a consequence of (3).

On your work sheet, translate each of these sentences into a sentence of the form:

if  then 

- (a) not (John is sick and John is not absent)

- (b) not ($a + 3 = 2$ and $a \neq -1$)

- (c) not ($b = 7$ and $b + 2 \neq 9$)

- (d) not ($a = 3$ and $a = 2$)

[RR]

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- (a) if John is sick then John is absent
- (b) if $a + 3 = 2$ then $a = -1$
- (c) if $b = 7$ then $b + 2 = 9$
- (d) if $a = 3$ then $a \neq 2$

Consider the sentence:

- (1) Cows do not eat hay or horses eat grass.

Can you write a conditional sentence [if then] which follows from (1)? We can if we can find a sentence of the form:

not (and not)

which follows from (1)

The sentence

- (2) not (cows eat hay and horses do not eat grass)

follows from (1). Since

- (3) if cows eat hay then horses eat grass

follows from (2) and (2) follows from (1), (3) is a consequence of (1).

On your work sheet, translate each sentence into a sentence of the form:

not (and not)

- (a) $a + 2 \neq 5$ or $a = 3$
- (b) Pete is not tall or Tom is smart
- (c) John is not sick or John is absent

[RR]

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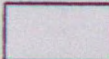
(a) $\text{not}(a + 2 = 5 \text{ and } a \neq 3)$

(b) $\text{not}(\text{Pete is tall and Tom is not smart})$

(c) $\text{not}(\text{John is sick and John is not absent})$

* * *

On your work sheet, translate each sentence into a sentence of the form:

if  then 

(a) $a + 2 \neq 5 \text{ or } a = 3$

(b) Pete is not tall or Tom is smart

(c) John is not sick or John is absent

[RR]

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- (a) if $a + 2 = 5$ then $a = 3$
- (b) if Pete is tall then Tom is smart
- (c) If John is sick then John is absent

* * *

We can also translate a conditional sentence into an alternation or a denial of a conjunction. For example, the conditional:

if $ab = 0$ then $a = 0$

can be translated to:

$ab \neq 0$ or $a = 0$

or can be translated to:

not($ab = 0$ and $a \neq 0$)

The ability to translate from one type of sentence to another will be a help to you as you continue your study of mathematics.

On your work sheet, translate each conditional sentence to an alternation and to the denial of a conjunction.

- (a) if $a = b$ then $a + c = b + c$
- (b) if $a = -b$ then $-a = b$
- (c) if $a > 0$ then $a^2 > 0$


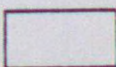
(a) if $a = b$ then $a + c = b + c$
 $a \neq b$ or $a + c = b + c$
 not ($a = b$ and $a + c \neq b + c$)

(b) if $a = -b$ then $-a = b$
 $a \neq -b$ or $-a = b$
 not ($a = -b$ and $-a \neq b$)

(c) if $a > 0$ then $a^2 > 0$
 $a \not> 0$ or $a^2 > 0$
 not ($a > 0$ and $a^2 \not> 0$)

* * *

Translate each sentence to a sentence of the form:

if  then 

(a) Anabru did not win or there was a celebration

(b) not ($a = b$ and $a + c \neq b + c$)

(c) It is not the case that the sun shines and flowers do not grow

(d) Horses do not eat hay or pigs squeal

(e) $a = 0$ if $3 \cdot a = 0$

[RR]

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- (a) if Anabru did win then there was a celebration
- (b) if $a = b$ then $a + c = b + c$
- (c) if the sun shines then flowers do grow
- (d) if horses eat hay then pigs squeal
- (e) if $3 \cdot a = 0$ then $a = 0$

* * *

Pete told Ann:

I will be at the game if I finish my homework by 6.

Ann heard later from Pete that he did not finish his homework by 6. She concluded that Pete would not be at the game. Was Ann's reasoning correct?

Let's translate Pete's statement:

- (1) I will be at the game if I finish my homework by 6.

into a conditional. The conditional:

- (2) If I finish my homework by 6 then I will be at the game.

is equivalent to Pete's statement (1). Ann had committed the fallacy of denying the antecedent. Pete had not told Ann what would happen if he did not finish his homework. Ann did not reason correctly about Pete.

Translate each of these sentences to a conditional:

- (a) $a + c = b + c$ if $a = b$
- (b) Horses fly if horses have wings
- (c) John is sick if John is absent

[RR]

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- (a) if $a = b$ then $a + c = b + c$
- (b) if horses have wings then horses fly
- (c) if John is absent then John is sick

* * *

Steve told Ann:

I will be at the game only if I finish my homework by 6.

Ann heard later from Steve that he had not finished his homework by 6. She concluded that Steve would not be at the game. Was Ann's reasoning correct?

Let's consider Steve's statement again. Steve did not finish his homework by 6. He had told Ann he would be at the game only if he finished his homework by 6. Since he did not finish his homework by 6, he was not at the game. Ann had reasoned correctly about Steve.

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Steve's statement:

- (1) I will be at the game only if I finish my homework by 6.

means the same as:

- (2) If I do not finish my homework by 6 then I will not be at the game.

And (2), by contraposition, gives:

if I am at the game then I did finish my homework.

The sentence:

$$a = 0 \text{ only if } 3 \cdot a = 0$$

translates to the conditional:

$$\text{if } a = 0 \text{ then } 3 \cdot a = 0$$

Translate each of these sentences into a sentence of the form:

if then

- (a) Horses eat hay if cows eat grass.
- (b) Horses eat hay only if cows eat grass.
- (c) $a + b = c$ if $a = -b$
- (d) $a + b = c$ only if $a = -b$

[RR]

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(a) If cows eat grass then horses eat hay

(b) If horses eat hay then cows eat grass

or

If cows do not eat grass then horses do not eat hay

(c) If $a = -b$ then $a + b = c$

(d) If $a + b = c$ then $a = -b$

or

If $a \neq -b$ then $a + b \neq c$

* * *

Translate each sentence to a conditional [if p then q]:

(a) John is not tall or Mary is smart

(b) Not ($a = b$ and $b \neq a$)

(c) Anabru won the game only if Tom played

(d) Mary is smart or John is not tall

(e) Harry is president if Sally is vice-president

(f) Anabru did not win or Tom played

(g) It is not the case that John is tall and Mary is not smart

(h) John is tall only if Mary is smart

- (a) if John is tall then Mary is smart
- (b) if $a = b$ then $b = a$
- (c) if Anabru won the game then Tom played
- (d) if John is tall then Mary is smart
- (e) if Sally is vice-president then Harry is president
- (f) if Anabru did win then Tom played
- (g) if John is tall then Mary is smart
- (h) if John is tall then Mary is smart

Turn to PAGE 268.

[RR]

[Page 268]

In your earlier work in mathematics you may have used sentences like:

$$(I) \quad a + c = b + c \text{ if and only if } a = b$$

The word 'and' in sentence (I) suggests a conjunction. Sentence (I) is an abbreviation for the conjunction:

$$(II) \quad (a + c = b + c \text{ if } a = b) \text{ and } (a + c = b + c \text{ only if } a = b)$$

Thus, the rules for conjunctions can be used with sentences like (I).

Now, let's look at each part of the conjunction (II). One part is:

$$(i) \quad a + c = b + c \text{ if } a = b$$

(i) is equivalent to the conditional:

$$(i') \quad \text{if } a = b \text{ then } a + c = b + c$$

Now, how about the other part?

$$(ii) \quad a + c = b + c \text{ only if } a = b$$

Suppose $a \neq b$. Since $a + c = b + c$ only if $a = b$, it follows that:

$$(ii') \quad \text{if } a \neq b \text{ then } a + c \neq b + c$$

But (ii') [by contraposition] is equivalent to:

$$(ii'') \quad \text{if } a + c = b + c \text{ then } a = b$$

Hence, the sentence (I) may be unabbreviated to:

$$(III) \quad (\text{if } a = b \text{ then } a + c = b + c) \text{ and } (\text{if } a + c = b + c \text{ then } a = b)$$

Sentence (I) is called a biconditional. Sentence (III) means the same as the biconditional (I). At times the form 'p if and only if q' is easier to use and other times the conjunction '(if p then q) and (if q then p)' is easier to use. Feel free to translate from one form to the other.

Turn to PAGE 269.

[RR]

[Page 269]

Since a biconditional, a sentence of the form:

p if and only if q

can be unabbreviated to a conjunction of the form:

(if p then q) and (if q then p)

we can use our inference patterns for conjunctions:

$$\frac{p \quad q}{p \text{ and } q}$$

$$\frac{p \text{ and } q}{p}$$

$$\frac{p \text{ and } q}{q}$$

when dealing with biconditionals. Here is an example:

Prove that:

$$\forall_x \forall_y \forall_z \quad x + z = y + z \text{ if and only if } x = y$$

is a theorem.

Proof:

- | | | |
|-----|---|--|
| | $\forall_x \forall_y \forall_z \text{ if } x + z = y + z \text{ then } x = y$ | |
| (1) | if $a + c = b + c$ then $a = b$ | UI |
| | | |
| | $\forall_x \forall_y \forall_z \text{ if } x = y \text{ then } x + z = y + z$ | |
| (2) | if $a = b$ then $a + c = b + c$ | UI |
| | | |
| (3) | (if $a + c = b + c$ then $a = b$)
and
(if $a = b$ then $a + c = b + c$) | (1), (2); $\frac{p \quad q}{p \text{ and } q}$ |
| | | |
| (4) | $a + c = b + c$ if and only if $a = b$ | (3); abbreviation |
| | | |
| (5) | $\forall_x \forall_y \forall_z \quad x + z = y + z \text{ if and only if } x = y$ | (1) - (4); TPP |

Whenever you can derive a conditional and also derive its converse, then you can derive the conjunction of these two conditionals, and the equivalent biconditional. Whenever you have proved the universal generalization of a biconditional you then make use of either part of the biconditional in a proof by using universal instantiation and one of the rules for conjunctions.

Turn to PAGE 270a.

[Page 270a]

[RR]

Here are three inference patterns for biconditionals:

$$(I) \frac{p \text{ if and only if } q}{\text{if } p \text{ then } q}$$

$$(II) \frac{p \text{ if and only if } q}{\text{if } q \text{ then } p}$$

$$(III) \frac{\text{if } p \text{ then } q \quad \text{if } q \text{ then } p}{p \text{ if and only if } q}$$

Note that these are similar to the inference pattern for conjunctions. There is another rule of inference for biconditionals. Here is an outline of a derivation which shows the use of this rule:

We can transform the sentence:

$$(1) (a + 7)(a - 3) = 0$$

into the equivalent sentence:

$$(2) a + 7 = 0 \text{ or } a - 3 = 0$$

and then transform (2) into the equivalent sentence:

$$(3) a = -7 \text{ or } a = 3$$

(2) follows from (1) by using the theorem:

$$\forall x \forall y \text{ if } xy = 0 \text{ then } x = 0 \text{ or } y = 0$$

What are the logical grounds for claiming that (3) and (2) are equivalent?

Turn to PAGE 8.

SKIP THIS PAGE

[RR]

[Page 270]

Using our principles and rules of reasoning, we can derive the two biconditionals:

$$(a) \quad a + 7 = 0 \text{ if and only if } a = -7$$

$$(b) \quad a - 3 = 0 \text{ if and only if } a = 3$$

We can now derive (3) from (2), (a), and (b) by using what is called the replacement rule for biconditionals. Consider (a) and (2):

$$(a) \quad a + 7 = 0 \text{ if and only if } a = -7$$

$$(2) \quad a + 7 = 0 \quad \text{or} \quad a - 3 = 0$$

We replace one side of (a) [$a + 7 = 0$] by its other side [$a = -7$] in (2) to obtain:

$$(2.1) \quad a = -7 \text{ or } a - 3 = 0$$

The replacement rule for biconditionals tells us that (2.1) follows from (a) and (2). Using (b) and (2.1) with the replacement rule for biconditionals we get:

$$(3) \quad a = -7 \text{ or } a = 3$$

Read the replacement rule for equations on page 75a. Using this as a pattern, write the replacement rule for biconditional sentences. Write the rule on your work sheet.

[RR]

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The Replacement Rule for Biconditionals

Given a biconditional sentence and another sentence, if one side of the biconditional sentence is replaced by the other side somewhere in the other sentence, the new sentence thus obtained is a consequence of the given sentences.

* * *

Use the replacement rule for biconditional sentences and complete these inferences:

(a) $\frac{a = -7 \text{ or } a - 3 = 0}{a - 3 = 0 \text{ if and only if } a = 3}$
?

(b) $\frac{\begin{array}{ccc} \text{Tom} & & \text{Tom} \\ \text{is} & \text{and} & \text{is} \\ \text{tall} & & \text{smart} \end{array}}{\begin{array}{c} \text{Tom is tall} \\ \text{if and only if} \\ \text{Tom plays basketball} \end{array}}$
?

(c) $\frac{\text{if } a^2 > 0 \text{ then } a > 0}{a \in P \text{ if and only if } a > 0}$
?

[RR]

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$$(a) \ a = -7 \text{ or } a = 3$$

(b)

Tom		Tom
plays	and	is
basketball		smart

$$c) \text{ if } a^2 > 0 \text{ then } a \in P$$

* * *

Sally noticed that the Gloxians did not have an AXIOM which corresponds with our principle:

$$\forall_x \ x \cdot 0 = 0$$

She wondered how they got along without such an AXIOM. Being a clever girl, Sally thought that maybe the sentence:

$$(I) \ \forall_x \ x \cdot 0 = 0$$

is a Gloxian theorem. If the sentence ' $\forall_x \ x \cdot 0 = 0$ ' is a consequence of the Gloxian AXIOMS then they do not need to include (I) as an AXIOM in their system.

The sentence:

$$(I) \ \forall_x \ x \cdot 0 = 0$$

is a Gloxian theorem.

On your work sheet, prove that (I) is a theorem.

[Hint: Use the AXIOMS:

$$\begin{aligned} \forall_x \ x \dot{+} 0 &= x \\ \forall_x \forall_y \forall_z \ (x \dot{+} y) \cdot z &= (x \cdot z) \dot{+} (y \cdot z) \end{aligned}$$

and derive the sentences

$$\begin{aligned} a \cdot a &= (a \cdot a) \dot{+} (0 \cdot a) \\ (a \cdot a) \dot{+} 0 &= (a \cdot a) \dot{+} (0 \cdot a) \end{aligned}$$

Then use the theorem:

$$\forall_x \forall_y \forall_z \text{ if } x \dot{+} y = x \dot{+} z \text{ then } y = z$$

[RR]

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Here is our derivation:

$$\forall_x \forall_y \forall_z (x \oplus y) \cdot z = (x \cdot z) \oplus (y \cdot z)$$

- (1) $(a \oplus 0) \cdot a = (a \cdot a) \oplus (0 \cdot a)$ UI
-
- (2) $a \oplus 0 = a$ $\forall_x x \oplus 0 = x$ UI
-
- (3) $a \cdot a = (a \cdot a) \oplus (0 \cdot a)$ (1), (2); RRE
-
- (4) $(a \cdot a) \oplus 0 = a \cdot a$ $\forall_x x \oplus 0 = x$ UI
-
- (5) $(a \cdot a) \oplus 0 = (a \cdot a) \oplus (0 \cdot a)$ (4), (3); RRE
-
- $\forall_x \forall_y \forall_z$ if $x \oplus y = x \oplus z$ then $y = z$
- (6) if $(a \cdot a) \oplus 0 = (a \cdot a) \oplus (0 \cdot a)$ then $0 = 0 \cdot a$ UI
-
- (7) $0 = 0 \cdot a$ (5), (6); MP
-
- (8) $a \cdot 0 = 0 \cdot a$ $\forall_x \forall_y x \cdot y = y \cdot x$ UI
-
- (9) $a \cdot 0 = 0$ (7), (8); RRE
-
- (10) $\forall_x x \cdot 0 = 0$ (1) - (9); TPP

If you have a different derivation, have your teacher check it.

* * *

The generalization:

$$(*) \quad \forall_x \forall_y \forall_z (x - y) \cdot z = x \cdot z - y \cdot z$$

can be derived by using some of the other principles. On your work sheet, write a derivation which shows that (*) is a theorem.

[RR]

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' $\forall_x \forall_y \forall_z (x - y) \cdot z = x \cdot z - y \cdot z$ ' is a theorem. It is a consequence of:

$$\begin{aligned}\forall_x \forall_y x - y &= x + -y \\ \forall_x \forall_y \forall_z (x + y) \cdot z &= x \cdot z + y \cdot z \\ \forall_x \forall_y (-x)y &= -(xy)\end{aligned}$$

If you have written a derivation of the sentence:

$$\forall_x \forall_y \forall_z (x - y) \cdot z = x \cdot z - y \cdot z$$

compare it with our derivation on the next page.

If you have not written a derivation of the sentence:

$$\forall_x \forall_y \forall_z (x - y) \cdot z = x \cdot z - y \cdot z$$

read this beginning of our derivation.

(1) $(a - b) \cdot c = (a - b) \cdot c$	$\forall_x x = x$	UI
(2) $a - b = a + -b$	$\forall_x \forall_y x - y = x + -y$	UI
(3) $(a - b) \cdot c = (a + -b) \cdot c$		(2), (1); RRE
(4)	$\forall_x \forall_y \forall_z (x + y) \cdot z = x \cdot z + y \cdot z$	UI

Now, you finish the derivation and then compare with our derivation on the next page.

[RR]

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- $$\forall_x x = x$$
- (1) $(a - b) \cdot c = (a - b) \cdot c$ UI
-
- $$\forall_x \forall_y x - y = x + -y$$
- (2) $a - b = a + -b$ UI
-
- (3) $(a - b) \cdot c = (a + -b) \cdot c$ (2), (1); RRE
-
- $$\forall_x \forall_y \forall_z (x + y) \cdot z = x \cdot z + y \cdot z$$
- (4) $(a + -b) \cdot c = a \cdot c + (-b) \cdot c$ UI
-
- (5) $(a - b) \cdot c = a \cdot c + (-b) \cdot c$ (4), (3); RRE
-
- $$\forall_x \forall_y (-x) \cdot y = -(x \cdot y)$$
- (6) $(-b) \cdot c = -(b \cdot c)$ UI
-
- (7) $(a - b) \cdot c = a \cdot c + -(b \cdot c)$ (6), (5); RRE
-
- $$\forall_x \forall_y x - y = x + -y$$
- (8) $a \cdot c - b \cdot c = a \cdot c + -(b \cdot c)$ UI
-
- (9) $(a - b) \cdot c = a \cdot c - b \cdot c$ (8), (7); RRE
-
- (10) $\forall_x \forall_y \forall_z (x - y) \cdot z = x \cdot z - y \cdot z$ (1) - (9); TPP

* * *

If you have a different derivation, have your teacher check it.

Turn to PAGE 276.

[RR]

[Page 276]

In these pages we have discussed some of the rules of reasoning and principles of logic used in proving theorems. Don't be disappointed if you feel that you do not have complete control over these rules. The ability to use them will grow as you think about applying them in writing proofs and in reading proofs. You will probably find it helpful to reread these pages at various times as you continue your study of mathematics.

As you continue your study of mathematics you will find that most people do not write complete proofs as we have done. Column proofs such as you have been writing are likely to be long, and tedious both to write and recall. You will probably develop a freer style for writing proofs of a kind which are called paragraph proofs. A paragraph proof is similar to the informal plan we wrote for some of our proofs. If you knew all about proving theorems, you would be able to write and understand paragraph proofs with no difficulty. However, you probably have many things to learn about proving theorems. Studying and writing column proofs or tree-chart proofs is a good way to learn these things. Even after you "graduate" to writing paragraph proofs, there will be times when you will find it convenient to expand part of a hard paragraph proof into a column proof or tree-chart proof just to make sure the reasoning is correct. Expanding part of someone else's paragraph proof into a column proof or tree-chart proof may also help you understand his reasoning, and possibly find a logical flaw in his reasoning.

This completes this book. Return this book and your work sheets to your teacher.

BASIC PRINCIPLES

Commutative principles for addition and multiplication

$$(cpa) \quad \forall_x \forall_y \quad x + y = y + x$$

$$(cpm) \quad \forall_x \forall_y \quad xy = yx$$

Associative principles for addition and multiplication

$$(apa) \quad \forall_x \forall_y \forall_z \quad (x + y) + z = x + (y + z)$$

$$(apm) \quad \forall_x \forall_y \forall_z \quad (xy)z = x(yz)$$

Distributive principle for multiplication over addition

$$(dpma) \quad \forall_x \forall_y \forall_z \quad (x + y)z = xz + yz$$

Distributive principle for multiplication over subtraction

$$(dpms) \quad \forall_x \forall_y \forall_z \quad (x - y)z = xz - yz$$

Principle for adding 0

$$(pa0) \quad \forall_x \quad x + 0 = x$$

Principle for multiplying by 1

$$(pml) \quad \forall_x \quad x \cdot 1 = x$$

Principle for multiplying by 0

$$(pm0) \quad \forall_x \quad x \cdot 0 = 0$$

Principle of opposites

$$(po) \quad \forall_x \quad x + -x = 0$$

Principle for subtraction

$$(ps) \quad \forall_x \forall_y \quad x - y = x + -y$$

APPENDIX B
LOGIC ACHIEVEMENT TEST

LOGIC EXAMINATION

This test consists of three parts. You will have 40 minutes for the entire test. There are three answer sheets, one for each part. Write your name and ID number at the top of each answer sheet. Do it now.

When your teacher tells you to begin, start with Part I. After finishing Part I, go on to Part II, and then to Part III. After going through the entire test, you may return to any questions you could not answer the first time through. Be sure you allow yourself time to work on each part of the test.

Turn to Part I.

UICSM-PIP
Logic Examination
Part I

Directions:

This part contains items designed to test your knowledge of terminology and conventions used in formal logic. Each item is followed by four answers, only one of which is correct. You are to mark your answers on the answer sheet. Do not write on the question sheets.

1. DO NOT TURN THE PAGE UNTIL YOU ARE TOLD TO BEGIN.
2. This is a 20-question, multiple-choice test.
3. There is only one correct choice for each question.

1. An instance of the generalization:

$$\forall_x \forall_y x + y = y + x$$

is: _____

- (A) $\forall_a \forall_b a + b = b + a$
 (B) $2 + a = a + 2$
 (C) $3 + (4 + 7) = 3 + (7 + 4)$
 (D) $(3 + 4) + 7 = (4 + 3) + 7$

2. A sentence is a theorem in an axiomatic system if and only if _____?

- (A) it is true (B) it is valid
 (C) it is a logical consequence of the axioms
 (D) it is a universal generalization

3. The contrapositive of 'if $a \neq 2$ then $a^2 \neq 4$ ' is: _____?

- (A) if $a \neq 2$ then $a^2 = 4$ (B) if $a^2 = 4$ then $a = 2$
 (C) if $a^2 = 4$ then $a = 2$ or $a = -a$ (D) if $a^2 \neq 4$ then $a \neq 2$

4. In the sentence:

$$\text{if } a = b \text{ then } a + 2 = b + 2$$

the sentence:

$$a = b$$

is the _____?

- (A) antecedent (B) consequent (C) converse (D) premiss

5. An inference scheme for contraposition is: _____?

- (A) $\frac{\text{if } p \text{ then } q}{\text{if } q \text{ then } p}$ (B) $\frac{\text{if } p \text{ then } q}{\text{not } (p \text{ and not } q)}$
 (C) $\frac{\text{if } p \text{ then } q}{\text{if not } q \text{ then not } p}$ (D) $\frac{\text{if } p \text{ then } q}{(\text{not } p) \text{ or } q}$

6. The sentence:

if $a = 2$ then $a + 3 = 5$

follows from:

$\forall x$ if $x = 2$ then $x + 3 = 5$

by _____?

(A) modus ponens

(B) replacement rule for equations

(C) test-pattern principle

(D) universal instantiation

7. Here is an inference:

$$\frac{\begin{array}{cc} \text{(i)} & \text{(ii)} \\ a = b & \text{if } a = b \text{ then } a + 1 = b + 1 \end{array}}{a + 1 = b + 1} \quad \text{(iii)}$$

The premiss(es) in this inference is (are) _____?

(A) only (i)

(B) (i) and (ii)

(C) (i) and (iii)

(D) (ii) and (iii)

8. An inference scheme for modus ponens is: _____?

(A) $\frac{\text{not } p \quad \text{if } q \text{ then } p}{\text{not } p}$

(B) $\frac{q \quad \text{if } q \text{ then } p}{p}$

(C) $\frac{p \quad \text{if } q \text{ then } p}{q}$

(D) $\frac{\text{not } q \quad \text{if } q \text{ then } p}{\text{not } p}$

9. The conditional:

if horses fly then hens lay eggs

is equivalent to:

(A) horses do not fly and hens lay eggs

(B) horses fly and hens do not lay eggs

(C) horses do not fly or hens lay eggs

(D) horses fly or hens do not lay eggs

10. Here is an inference:

$$\frac{a \neq b \quad \text{if } a \neq b \text{ then } a + c = b + c}{a + c = b + c}$$

The rule of reasoning which justifies this inference is _____?

- (A) conditionalizing (B) contraposition
(C) modus ponens (D) modus tollens

11. The converse of:

$$\text{if } ac = bc \text{ then } a = b$$

is: _____?

- (A) if $ac \neq bc$ then $a \neq b$
(B) if $a \neq b$ then $ac \neq bc$
(C) if $a = b$ then $ac = bc$
(D) if $bc = ac$ then $b = a$

12. Each inference which is justified by one of the accepted rules of reasoning is _____?

- (A) a proof (B) a theorem
(C) true (D) valid

13. From the sentence:

Cows eat grass.

the sentence:

If pigs are sweet then cows eat grass.

follows by _____?

- (A) conditionalizing (B) contraposition
(C) modus ponens (D) modus tollens

14. From the sentences:

- (i) today is not Friday
and (ii) if fish swim upstream then today is Friday

Mary concludes: fish do not swim upstream

Her reasoning is justified by _____ ? _____.

- (A) conditionalizing (B) modus ponens
(C) modus tollens (D) no rule of reasoning

15. The sentence:

$$a^2 = 4 \text{ only if } a = 2$$

is equivalent to:

- (A) if $a = 2$ then $a^2 = 4$ (B) $a^2 = 4$
(C) if $a^2 = 4$ then $a = 2$ (D) $a = 2$

16. A contradictory of:

$$2 = 1 + 1 \quad \text{and} \quad 5 > 7$$

is: _____ ? _____

- (A) $2 \neq 1 + 1$ and $5 \not> 7$ (B) $2 = 1 + 1$ and $5 \not> 7$
(C) $2 \neq 1 + 1$ or $5 \not> 7$ (D) $2 = 1 + 1$ or $5 \not> 7$

17. The only listed sentence which is a universal generalization

is: _____ ? _____

- (A) $x + y = y + x$
(B) $\forall_x \exists_y x + y = 0$
(C) $\exists_x \forall_y y \cdot x = 0$
(D) Some numbers are prime

18. Consider this inference scheme:

$$\frac{\text{not } p \quad \text{if } p \text{ then } q}{\text{not } q}$$

To get an inference from the inference scheme, you should substitute _____ ? _____ for 'p' and 'q'.

- | | |
|-----------------|---------------|
| (A) numerals | (B) variables |
| (C) pronumerals | (D) sentences |

19. The sentence:

$$3x - 5 = 0 \quad \text{if and only if} \quad x = 2$$

is a(n) _____ ? _____.

- | | |
|----------------------------|----------------------------|
| (A) biconditional sentence | (B) conditional sentence |
| (C) true sentence | (D) unconditional sentence |

20. If a universal sentence has a counter-instance, then we say that the sentence is _____ ? _____.

- | | |
|-------------------|--------------------------|
| (A) a theorem | (B) a generalization |
| (C) not a theorem | (D) not a generalization |

Go on to Part II.

Part II

This part is designed to test your ability to correctly complete inferences. Each item is an incomplete derivation. Your job is to write a sentence which correctly completes the derivation.

Write your answers on your answer sheet.

1. $\frac{\quad ? \quad \text{if cows give milk then today is Tuesday}}{\text{today is Tuesday}}$
2. $\frac{\text{not } (x \neq 2)}{?}$
3. $\frac{\quad ? \quad a + 3 = b}{2 + 3 = b}$
4. $\frac{\text{Albany is the capital of New York} \quad \text{Herb lives in Albany}}{?}$
5. $\frac{\text{if } ab = 0 \text{ then } b = 0 \quad b \neq 0}{?}$
6. $\frac{\text{if } a = 2 \text{ then } a + c = 5 \quad ?}{a \neq 2}$
7. $\frac{a = 0 \text{ or } b = 0 \quad \text{if } a = 0 \text{ then } ab = 0 \quad \text{if } b = 0 \text{ then } ab = 0}{?}$
8. $\frac{\frac{\forall_x \forall_y \quad xy = yx}{a3 = 3a} \quad \frac{\forall_x \quad x = x}{2(a3) = 2(a3)}}{?}$
 $\frac{\quad}{\forall_x \quad 2(x3) = 2(3x)}$
9. $\frac{\forall_x \quad 2x + 3x = 5x}{?}$
10. $\frac{\text{if } (a > 0 \text{ and } b > c) \text{ then } ab > ac \quad ab \not> ac}{?}$
11. $\frac{\text{ABCD is a rectangle} \quad \text{if } ?}{\text{ABCD is a square}}$
12. $\frac{\text{not } [(a = 2) \text{ or } (a = 1)]}{?}$

13. $\frac{\text{if ? then ?}}{\text{if stars are bright then comets are cute}} \quad \text{if birds swim then comets are cute}$
14. $\frac{\begin{array}{l} * \\ a = 2 \quad \text{if } a = 2 \text{ then } 2a + 5 = 9 \\ \hline 2a + 5 = 9 \quad \text{if } 2a + 5 = 9 \text{ then } 2a - 4 = 0 \\ \hline 2a - 4 = 0 \\ \hline ? \end{array}}{\quad} *$
15. $\frac{\text{if } a + b = 0 \text{ then } a = 0}{\text{if ? then ?}} \quad \frac{\text{if } a = 0 \text{ then } ab = 0}{\text{then ?}}$
16. $\frac{\text{if } a \neq 2 \text{ then } a + 3 = 5}{\text{if ? then ?}}$
17. $\frac{a + a = 0 \quad \text{if and only if} \quad a = 0}{\text{if } a + a = 0 \text{ then } a = 0 \text{ and} \quad ?}$
18. $\frac{?}{\text{if horses fly then cows give milk}}$
19. $\frac{\text{cows are red}}{?} \quad \frac{\text{if (bats are blind or cows are red) then barns are blue}}{\text{barns are blue}}$
20. $\frac{\text{if all men are mortal then Socrates is mortal}}{\text{if Socrates is not mortal then} \quad ?}$

Go on to Part III.

Part III

Each of these items is an example of inferring a conclusion from certain premisses. If you think the reasoning is correct, put a check mark, '✓', in the column headed 'Valid' on your answer sheet. If you think the reasoning is not correct, put a check mark in the column headed 'Invalid'.

1. Harry went to the game and Mary stayed home Harry went to the game
 Mary stayed home
2. All animals are cows. Since Sam is not a cow, it follows that Sam is not an animal.
3. Suppose Joe is a man and not all men are tall. It follows that Joe is not tall.
4. $\forall x \forall y$ if $x + y = 0$ then $y = -x$. Also, $-5 = -5$. Hence, $-5 + 5 = 0$.
5. Suppose all fruit are green and prunes are fruit. So, prunes are green.
6. If $a + b = 0$ then $a = 0$. But $a \neq 0$. Hence, $a + b \neq 0$.
7.
$$\frac{5 = 6}{6 + (9 \times 10) = 5 \times (9 + 10)} \quad \begin{array}{l} 5 + (9 \times 10) = 5 \times (9 + 10) \\ 6 + (9 \times 10) = 5 \times (9 + 10) \end{array}$$
8. If I ride the bus or walk, I get tired. I do not get tired if I drive. On Tuesday I didn't drive. So, I rode the bus or walked.
9. Suppose $\triangle ABC$ is isosceles. If triangle ABC is isosceles then $\triangle ABC$ is equilateral. If $\triangle ABC$ is equilateral then $AB = BC = AC$. So, if $\triangle ABC$ is isosceles, then $AB = BC = AC$.
10. If roses are red and violets are blue then horses eat grass. But roses are red. Hence, if violets are blue then horses eat grass. So, we may conclude that if horses do not eat grass then violets are not blue.

APPENDIX C

PARTICIPATING COORDINATORS AND TEACHERS

Jefferson High, Denver, Colorado

Mr. John Bradford (Coordinator)
Miss Judith Sandstrom

Wheat Ridge Junior High, Wheat Ridge, Colorado

Mrs. Donna Hays

Pascack Valley High, Hillsdale, New Jersey

Miss Maureen Jordan (Coordinator and Teacher)
Mr. Andrew Guerriero
Miss Lorraine Alberto

Boulder City Junior-Senior High, Boulder City, Nevada

Mr. James Nelson (Coordinator and Teacher)
Mr. Richard Whitney
Mr. Dale Foreman

Central Catholic High, Portland, Oregon

Sister Mary Eleanor Christine (Coordinator and Teacher)
Sister Mary Aquinas
Sister Mary Denise
Father Donald McHugh

VITA

William Thomas Hale was born on April 5, 1928, in Monson, Massachusetts, where he attended elementary, junior and senior high school. After serving three years in the United States Army, he attended Bates College and received his A. B. degree in 1953. In 1954 he received his Ed. M. degree from the University of Illinois. After two years of teaching mathematics at Lowville Academy and Central School in Lowville, N. Y., he returned to the University of Illinois in 1956 as a member of the staff of the University of Illinois Committee on School Mathematics and as a mathematics teacher at University High School, Urbana, Illinois. While on the staff of the UICSM he has served as assistant director of the mathematics project, associate director of summer institutes for secondary school mathematics teachers, and chairman of the mathematics department of University High School. In December, 1961, his article "UICSM's Decade of Experimentation" was published in The Mathematics Teacher.